Portfolio Choice with a Large Number of Assets: Jumps and Diversification*

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ABSTRACT

We analyze the portfolio selection problem of an investor facing both Brownian and jump risks. By decomposing the two types of risks on a well-chosen basis, we provide a new methodology for determining the optimal solution in closed form, up to a constant. We show that the optimal solution is for the investor to focus on controlling his exposure to the jump risk, while exploiting differences in the asset returns diffusive characteristics in the orthogonal space. We then examine the solution to the portfolio problem as the number of assets available to the investor increases, and study the asymptotic distribution of the investor's wealth and optimal portfolio.

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I. Introduction

Economists have long been aware of the potential benefits of international diversification, while at the same time noting that the portfolios held by actual investors typically suffer from a home bias effect (see e.g., Grubel (1968), Levy and Sarnat (1970), Solnik (1974), Grauer and Hakansson (1987)). One possible explanation is due to the risk of contagion across markets in times of crisis (see e.g., Claessens and Forbes (2001), Longin and Solnik (2001), Ang and Chen (2002), Bae, Karolyi, and Stulz (2003) and Hartmann, Straetmans, and de Vries (2004).) A natural way to capture contagion mathematically is by introducing jumps. Jumps of correlated sign will generate the type of asymmetric correlation across markets that is often used to justify the home bias exhibited by investors' portfolios. Namely, when a downward jump occurs, negative returns tend to be experienced simultaneously across most markets, which then results in a high positive correlation in bear markets. When no jump occurs, the only source of correlation is that generated by the driving Brownian motions and will typically be much lower.

Studying the impact of jumps on portfolio choice has a long history, going back to Merton (1971), who first studied a continuous-time consumption-portfolio problem. Many papers have considered the portfolio problem, either in the simple one-period Markowitz setting or in the more complex Merton setting, when asset returns are generated by jump processes, for instance Poisson processes, stable processes or more general Lévy processes. Early papers include Aase (1984), Jeanblanc-Picque and Pontier (1990) and Shirakawa (1990). More recently, see Han and Rachev (2000) for a study of the Markowitz one-period mean-variance problem when asset returns follow a stable-Paretian distribution; Kallsen (2000) for a study of the continuous-time utility maximization in a market where risky security prices follow Lévy processes, and a solution (up to integration) for power, logarithmic and exponential utility using the duality or martingale approach; Choulli and Hurd (2001) give solutions up to constants of the primal and dual Merton portfolio optimization problem for the exponential, power and logarithmic utility functions when a risk-free asset and a exponential Lévy stock are the investment assets; Liu, Longstaff, and Pan (2003) study the implications of jumps in both prices and volatility on investment strategies when a risk-free asset and a stochastic-volatility jump-diffusion stock are the available investment opportunities; Emmer and Klüppelberg (2004) study a continuous-time mean-variance problem with multiple assets; Madan (2004) derives the equilibrium prices in an economy with single period returns driven by exposure to explicit non-Gaussian systematic factors plus Gaussian idiosyncratic components.

The potential role of jumps in generating contagion across markets, and hence limiting the benefits of diversification, has been investigated by Das and Uppal (2004), who evaluate the effect on portfolio choice of systemic risk, defined as the risk from infrequent events that are highly correlated across a large number of assets. They find that systemic risk reduces the gains from diversifying across a range of assets, and makes leveraged portfolios more susceptible to large losses. Upon calibrating their model to index returns, they find that the loss from the reduction in diversification is not substantial. Ang and Bekaert (2002) consider a two-regime model in a discrete-time setting, one with low correlations and low volatilities, and one with higher correlations, higher volatilities, and lower conditional means. They find that the existence of a high-volatility bear market regime does not negate the benefits of international diversification for an investor who dynamically rebalances his portfolio in response to regime switches.

Unfortunately, when jumps are present, the portfolio choice problem does not have a closed-form solution. With n assets, one must solve numerically an $n-$dimensional nonlinear equation (as for instance in Das and Uppal (2004).) With more efficient global markets, capital flows and a considerably larger number of available assets to invest in, an investor has more investment opportunities than ever before. We would certainly like to be able to consider models with a large number of assets n, where the characteristic Lévy triple has specific forms of n dependence. This is difficult to do using existing methodologies.

Our contribution to the solution method is to show that by selecting a *well-chosen basis* in the space spanned by the jump vector and the covariance matrix of returns, we can obtain closed-form solutions up to a constant, irrespectively of the number of assets. In our model, the structure of the Brownian volatility matrix is taken to reflect the existence of one or more economic sectors, each sector comprising a large number of related companies (or countries). The structure of the jump measure is taken to reflect the existence of a "contagion risk factor" which generates highly correlated negative returns across the range of assets. In the case where the Brownian covariance matrix corresponds to only one economic sector, then we are able to reduce the problem to a single scalar constant to be found. In the more general case where the economy consists of m sectors or regions of the world (each consisting of k firms or countries), we reduce the problem to an $m-$dimensional constant vector.

From a practical standpoint, this greatly simplifies the implementation of the optimal solution. This *dimensional reduction* is particularly important when the number of assets is large, that is when $n \to \infty$; in the $m-$sector case this corresponds to a fixed number of sectors or regions of the world (m fixed) and a growing number of firms in each sector or countries in each region ($k \to \infty$), with the total number of assets given by $n = mk$. From a theoretical standpoint, this quasi-closed form solution allows us to do explicit comparative statics, and give a precise characterization of the optimal portfolio and resulting wealth dynamics. In particular, we are able to distinguish between the optimal portfolio positions in the space spanned by the jump risk (which the investor will attempt to limit) and those in the orthogonal space (where the investor will seek to exploit the opportunities arising from the traditional risk-return trade-off.)

Our results show that if there is enough cross-sectional variability in the expected excess returns then the investor will place a linearly increasing amount of wealth in the risky assets as the number of assets n grows. This, in turn, leads to increasing expected return and volatility of the portfolio value, both growing linearly in the number of assets. And the optimal policy is to control the exposure to contagion jumps by keeping it bounded as the number of assets grows. As a result, the exposure to jumps is dwarfed by the exposure to diffusive risk asymptotically in n. Indeed, the additional investments in the risky assets are entirely in the direction that is orthogonal to the jump risk; they are all achieved with zero net additional exposure to the jump risk. In other words, the optimal investment policy is to control the overall exposure to jump risk, and then exploit, in the orthogonal space, any perceived differences in expected returns and diffusive variances. But in the special case where the expected excess returns have little variability in the orthogonal space, the opportunities for diversification effects are weak. The optimal portfolio in this case is not much better protected against contagion than a nondiversified portfolio.

The rest of the paper is organized as follows. In Section II, we present our model of asset returns, and examine the investor's portfolio selection problem. In Section III, we consider a one sector economy where the n risky assets have the same jump size, we construct the basis that allows us to reduce the problem to a one dimensional one, derive the optimal portfolio weights, and analyze the asymptotic behavior of the optimal portfolio. In Section IV, we consider an m-sector economy where sectors have different jump sizes and show how to solve the optimal portfolio problem up to an $m-$dimensional constant vector. In Section V, we study the situation where the risky assets are subject to jumps but the investor assumes that the returns are driven exclusively by Brownian motions but with first and second moments that are adjusted to reflect the presence of the jumps; in other words, the investor makes a partial adjustment to account for the jumps, by lumping them together with Brownian volatility. An example illustrating our methodology in the case of worldwide asset allocation is given in Section VI. Extensions, limitations of our theory, and conclusions are in Section VII.

II. The Portfolio Selection Model

A. Asset Return Dynamics

Like most of the above-mentioned literature, our paper focuses on Merton's problem of maximizing expected utility of terminal wealth by investing in a set of risky assets. That is, we select the amounts to be held in the n risky assets and the riskless asset at times $t \in [0, T]$. The available investment opportunities consist of a riskless asset with price $S_{0,t}$ and n risky assets with prices $\mathbf{S}_t = [S_{1,t}, \ldots, S_{n,t}]'$. These follow the exponential Lévy dynamics

$$\frac{dS_{0,t}}{S_{0,t}} = rdt, \tag{1}$$

$$\frac{dS_{i,t}}{S_{i,t-}} = (r + R_i)\, dt + \sum_{j=1}^{n} \sigma_{i,j} dW_{j,t} + J_i Z_t dN_t, \ i = 1, \ldots, n \tag{2}$$

with a constant rate of interest $r \geq 0$. N_t is a scalar Poisson process with constant intensity λ, $\mathbf{W}_t = [W_{1,t}, \ldots, W_{n,t}]'$ is an $n-$dimensional standard Brownian motion, and $J_i Z_t$ is the random jump amplitude. Z_t is a scalar random variable with probability measure $\nu(dz)$ on $[0, 1]$. The economy-wide jump amplitude Z_t is scaled on an asset-by-asset basis by the scaling factor $J_i \in [-1, +\infty)$. We assume that the individual Brownian motions, the Poisson jump and the random variables Z are mutually independent. The quantities R_i, σ_{ij}

and jump scaling factors J_i are constant parameters: We write $\mathbf{R} = [R_1, \ldots, R_n]'$, $\mathbf{J} = [J_1, \ldots, J_n]'$, and assume that

$$\sigma = \begin{pmatrix} \sigma_{1,1} & \cdots & \sigma_{1,n} \\ \vdots & \ddots & \vdots \\ \sigma_{n,1} & \cdots & \sigma_{n,n} \end{pmatrix} \tag{3}$$

is a nonsingular matrix. The expected excess returns and the return covariance matrix over short time intervals are given by

$$\hat{R} = R + \lambda J \bar{Z} \tag{4}$$

$$\hat{\Sigma}_{ij} = (\sigma\sigma')_{ij} + \lambda J_i J_j \bar{Z}^2 \tag{5}$$

where $\bar{Z} = \int_0^1 z\nu(dz)$, $\bar{Z}^2 = \int_0^1 z^2\nu(dz)$.

B. Wealth Dynamics

Let $\pi_{0,t}$ denote the amount invested at time t in the riskless asset and $\pi_t = [\pi_{1,t}, \ldots, \pi_{n,t}]'$ denote the vector of amounts invested in each of the n risky assets; these are assumed to be adapted cáglád processes. In the absence of any income derived outside his investments in these assets and of consumption, the investor's wealth, starting with the initial endowment X_0, is $X_t = \sum_{i=0}^n \pi_{i,t}$, so that

$$\pi_{0,t} = X_t - \sum_{i=1}^n \pi_{i,t} = X_t - \pi_t'\mathbf{1}. \tag{6}$$

where $\mathbf{1}$ is the $n-$vector $\mathbf{1} = [1, ..., 1]'$. The investor's wealth dynamics are given by

$$\begin{aligned} dX_t &= \pi_{0,t}\frac{dS_{0,t}}{S_{0,t-}} + \sum_{i=1}^n \pi_{i,t}\frac{dS_{i,t}}{S_{i,t-}} \\ &= \left(X_t - \pi_t'\mathbf{1}\right) r dt + \sum_{i=1}^n \pi_{i,t}\frac{dS_{i,t}}{S_{i,t-}}. \end{aligned} \tag{7}$$

The time t wealth valued in time 0 dollars is $Y_t = e^{-rt}X_t$, and follows

$$\begin{aligned} dY_t &= -re^{-rt}X_t dt + e^{-rt}dX_t \\ &= -e^{-rt}\pi_t'\mathbf{1}rdt + e^{-rt}\sum_{i=1}^n \pi_{i,t}\frac{dS_{i,t}}{S_{i,t-}} \\ &= e^{-rt}\pi_t'\mathbf{R} + e^{-rt}\pi_t'\sigma d\mathbf{W}_t + e^{-rt}\pi_t'\mathbf{J}Z_t dN_t. \end{aligned} \tag{8}$$

Let $\omega_t = e^{-rt}\pi_t$ denote the vector of time t dollar amounts invested in the risky assets, but expressed in time 0 dollars. We have

$$dY_t = \omega_t' \mathbf{R} dt + \omega_t' \sigma d\mathbf{W}_t + \omega_t' \mathbf{J} Z_t dN_t. \tag{9}$$

C. Optimal Portfolio Choice

The investor's problem at time t is to pick the portfolio vector process $\{\omega_s\}_{t \leq s \leq T}$ which maximizes the expected utility of terminal wealth,

$$V(Y_t, t) = \max_{\{\omega_s; t \leq s \leq T\}} E_t[U(Y_T)] \tag{10}$$

subject to the dynamics of his discounted wealth (9), and with Y_t given.

Using stochastic dynamic programming and the appropriate form of Itô's lemma for semi-martingale processes, the Hamilton-Jacobi-Bellman equation characterizing the optimal solution to the investor's problem is:

$$0 = \frac{\partial V(Y_t, t)}{\partial t} + \max_{\{\omega_t\}} \left\{ \frac{\partial V(Y_t, t)}{\partial Y} \omega_t' \mathbf{R} + \frac{1}{2} \frac{\partial^2 V(Y_t, t)}{\partial Y^2} \omega_t' \Sigma \omega_t \right. \tag{11}$$
$$\left. + \ \lambda \int_0^1 \left[V\left(Y_t + \omega_t' \mathbf{J} z, t\right) - V(Y_t, t) \right] \nu(dz) \right\}.$$

where $\Sigma = \sigma\sigma'$.

Consider an investor with exponential utility, $U(y) = -\exp(-\gamma y)$ with CARA coefficient $\gamma > 0$. We can look for a solution to (11) in the form

$$V(y, t) = -e^{K(T-t)} e^{-\gamma y} \tag{12}$$

where K is constant, so that

$$\frac{\partial V(y, t)}{\partial t} = -KV(y, t), \quad \frac{\partial V(y, t)}{\partial y} = -\gamma V(y, t), \quad \frac{\partial V(y, t)}{\partial y} = \gamma^2 V(y, t). \tag{13}$$

Then (11) reduces to

$$0 = -KV(Y_t, t) + \max_{\{\omega_t\}} \left\{ -\gamma V(Y_t, t) \omega_t' \mathbf{R} + \frac{1}{2} \gamma^2 V(Y_t, t) \omega_t' \Sigma \omega_t \right.$$
$$\left. + \ \lambda \int_0^1 \left[e^{-\gamma \omega_t' \mathbf{J} z} V(Y_t, t) - V(Y_t, t) \right] \nu(dz) \right\}. \tag{14}$$

that is

$$0 = -K + \min_{\{\omega_t\}} \left\{ -\gamma\omega_t'\mathbf{R} + \frac{1}{2}\gamma^2\omega_t'\Sigma\omega_t + \lambda \int_0^1 \left[e^{-\gamma\omega_t'\mathbf{J}z} - 1 \right] \nu\,(dz) \right\}. \tag{15}$$

after division by $V(Y_t, t)$ (note that max becomes min as a result of $V(Y_t, t) < 0$).

It is important to note that the objective function

$$g(\omega) = -\gamma\omega'\mathbf{R} + \frac{1}{2}\gamma^2\omega'\Sigma\omega + \lambda \int_0^1 \left[e^{-\gamma\omega'\mathbf{J}z} - 1 \right] \nu\,(dz)\,, \tag{16}$$

is time independent, strictly convex, goes to $+\infty$ in all directions, and hence always has a unique time independent minimizer which is proportional to γ^{-1}:

$$\omega^* = \arg\min_{\{\omega_t\}} g(\omega_t). \tag{17}$$

In the pure diffusive case, $\lambda = 0$ and we obtain of course the familiar solution

$$\omega^* = \frac{1}{\gamma}\Sigma^{-1}\mathbf{R}. \tag{18}$$

D. Lévy Processes

To characterize the resulting wealth dynamics along an optimal path, it is worthwhile at this point to place this result in the context of general Lévy processes. An n dimensional Lévy process $\mathbf{L}_t$ is specified by its "characteristic triple" $(\mathbf{b}, \mathbf{c}, \mu)$ where $\mathbf{b} \in \mathbb{R}^n$ is the drift or mean return vector, $\mathbf{c} \in \mathbb{R}^{n\times n}$ is the diffusion matrix, or local variance of the continuous part of $\mathbf{L}_t$, and μ is the jump or Lévy measure on $\mathbb{R}^n$, which satisfies

$$\int_{\mathbb{R}^n} \left(1 \wedge \|\mathbf{x}\|^2\right) \mu(d\mathbf{x}) < \infty.$$

The characteristic function of $\mathbf{L}_t$ is given by the Lévy-Khintchine formula

$$E(e^{i\mathbf{u}'\mathbf{L}_t}) = \exp\left(t \left(i\mathbf{u}'\mathbf{b} - \frac{\mathbf{1}}{2}\mathbf{u}'\mathbf{c}\mathbf{u} + \int_{\mathbb{R}^n\setminus\{0\}} \mu(d\mathbf{x}) \left(e^{i\mathbf{u}'\mathbf{x}} - 1 - i\mathbf{u}'\mathbf{h}\,(\mathbf{x}) \right) \right) \right) \tag{19}$$

for $\mathbf{u} \in \mathbb{R}^n$and $\mathbf{h}\,(\mathbf{x})$ is a truncation function which, because we are dealing with finite intensity measures, can be set to zero [see e.g., Chapter II.2 in Jacod and Shiryaev (2003)]. The stochastic differential equation for $\mathbf{L}_t$ written in terms of its characteristics is

$$d\mathbf{L}_t = \mathbf{b}dt \quad + \mathbf{c}^{1/2}d\mathbf{W}_t + \int_{\mathbb{R}^n} \mathbf{x}\; N^{(\mu)}(d\mathbf{x}, dt) \tag{20}$$

where $\mathbf{c}^{1/2}$ is a matrix square root satisfying $\mathbf{c}^{1/2}(\mathbf{c}^{1/2})' = \mathbf{c}$, and $N^{(\mu)}$ is called the Poisson random measure associated with the Lévy measure μ.

We can identify the right hand side of equation (2) as the dynamics of a Lévy process with triple $(r + \mathbf{R}, \sigma\sigma', \mu)$ where $\mu(d\mathbf{l}) = \lambda\nu(dz)$ with $\mathbf{l} = \mathbf{J}z$ a measure on a line segment in the direction of $\mathbf{J}$ in $\mathbb{R}^n$. We say that $\mathbf{S}_t$ itself has exponential Lévy dynamics, meaning that each component satisfies $dS_{i,t}/S_{i,t-} = dL_{i,t}$ where $\mathbf{L}_t = [L_{1,t}, \ldots, L_{n,t}]'$ follows an SDE of the type (20). Equation (9) shows that for Merton's problem with CARA utility the discounted optimal wealth process Y_t^* achieved by picking the constant portfolio amounts ω^* is itself a one dimensional arithmetic Lévy process whose characteristic triple is $(\omega'\mathbf{R}, \omega'\sigma\sigma'\omega, \mu)$ where $\mu(dy) = \lambda\nu(dz)$ with $y = \omega'\mathbf{J}z$. The investor keeps constant amounts proportional to γ^{-1} in each risky asset, and all trading losses and gains are taken from or put into the money market account.

III. Optimal Portfolio in the Presence of Jumps in a One Sector Economy

A. *Homogeneous Assets*

To begin, we consider the simplest possible case, where the n risky assets have the same jump size and expected excess return characteristics

$$\mathbf{J} = \bar{J}\mathbf{1} \tag{21}$$

$$\mathbf{R} = \bar{R}\mathbf{1} \tag{22}$$

with $\bar{J}$ and $\bar{R}$ scalars. To fix ideas, let us assume that $\bar{J} < 0$ in order to capture the downward risk inherent in the types of contagion jumps we are concerned about. As for Σ, we assume the one factor structure

$$\Sigma = v^2 \begin{pmatrix} 1 & \rho & \cdots \\ \rho & \ddots & \rho \\ \cdots & \rho & 1 \end{pmatrix} \tag{23}$$

where $v^2 > 0$ is the variance of the returns generated by the diffusive risk, and $0 < \rho < 1$ is their common correlation coefficient.

The key to characterizing the optimal portfolio solution in this simple situation is to decompose both Σ and ω on a well chosen basis, consisting in this case of the $n-$vector $\mathbf{1}$ and its orthogonal hyperplane. Specifically, the spectral decomposition of the Σ matrix is:

$$\Sigma = \underbrace{\kappa_1 \frac{1}{n}\mathbf{1}\mathbf{1}'}_{\bar{\Sigma}} + \underbrace{\kappa_2 \left(\mathbf{I} - \frac{1}{n}\mathbf{1}\mathbf{1}'\right)}_{\Sigma^{\perp}} \tag{24}$$

where $\mathbf{I}$ denotes the $n \times n$ identity matrix and

$$\kappa_1 = v^2 + v^2 (n-1) \rho \tag{25}$$

$$\kappa_2 = v^2 (1 - \rho) \tag{26}$$

are the two distinct eigenvalues of Σ, κ_1 with multiplicity 1 and eigenvector $\mathbf{1}$ and κ_2 with multiplicity $n-1$.

Let us decompose the portfolio weight vector ω according to

$$\omega = \bar{\omega}\mathbf{1} + \omega^{\perp}, \tag{27}$$

where $\bar{\omega}$ is scalar and $\omega^{\perp}$ is an $n-$vector orthogonal to $\mathbf{1}$. Then, from (17), the optimal $\bar{\omega}^*$ and $\omega^{\perp *}$ must satisfy

$$\left(\omega^{\perp *}, \bar{\omega}^*\right) = \arg \min_{\{\omega^{\perp}, \bar{\omega}\}} \left\{ -\gamma n \bar{\omega} \bar{R} + \frac{1}{2}\gamma^2 n \bar{\omega}^2 \kappa_1 + \lambda \int_0^1 \left[e^{-\gamma n \bar{\omega} \bar{J} z} - 1 \right] \nu(dz) + \frac{1}{2}\gamma^2 \omega^{\perp\prime} \Sigma^{\perp} \omega^{\perp} \right\}. \tag{28}$$

which separates into an optimization problem for $\omega^{\perp}$ and a separate one for the scalar $\bar{\omega}$:

$$\left(\omega^{\perp *}, \bar{\omega}^*\right) = \arg \min_{\{\omega^{\perp}, \bar{\omega}\}} \left\{ g^{\perp}(\omega^{\perp}) + \bar{g}(\bar{\omega}) \right\} \tag{29}$$

where

$$g^{\perp}(\omega^{\perp}) = \frac{1}{2}\gamma^2 \omega^{\perp\prime} \kappa_2 \left(\mathbf{I} - \frac{1}{n}\mathbf{1}\mathbf{1}'\right) \omega^{\perp} \tag{30}$$

$$\bar{g}(\bar{\omega}) = -\gamma n \bar{\omega} \bar{R} + \frac{1}{2}\gamma^2 n^2 \bar{\omega}^2 \kappa_1 \frac{1}{n} + \lambda \int_0^1 \left(e^{-\gamma n \bar{\omega} \bar{J} z} - 1 \right) \nu(dz). \tag{31}$$

The optimal solution for $\omega^{\perp}$ in this case is obviously

$$\omega^{\perp *} = \mathbf{0}. \tag{32}$$

As for the optimal solution for $\bar{\omega}$, with the change of variable $\varpi_n = n\bar{\omega}$, we see that

$$\varpi_n^* = \arg\min_{\{\varpi_n\}} \left\{ -\gamma \varpi_n \bar{R} + \frac{1}{2}\gamma \varpi_n^2 \kappa_1 / n + \lambda \int_0^1 \left[e^{-\gamma \varpi_n \bar{J} z} - 1 \right] \nu(dz) \right\}. \tag{33}$$

Letting $n \to \infty$, we have that $\kappa_1/n \to v^2\rho$ and so $\varpi_n^* \to \varpi_\infty^*$ where

$$\varpi_\infty^* = \arg\min_{\{\varpi_\infty\}} \left\{ -\gamma \varpi_\infty \bar{R} + \frac{1}{2}\gamma^2 \varpi_\infty^2 v^2 \rho + \lambda \int_0^1 \left[e^{-\gamma \varpi_\infty \bar{J} z} - 1 \right] \nu(dz) \right\}. \tag{34}$$

The optimal portfolio choice is characterized by $\omega^* = \bar{\omega}^* \mathbf{1} + \omega^{\perp *} = \bar{\omega}^* \mathbf{1} = \varpi_n^* \mathbf{1}/n$ and an investor who selects this optimal portfolio will achieve a discounted wealth process Y_t which follows an arithmetic Lévy process with characteristic triple (b, c, μ) given by

$$b = \omega^{*\prime}\mathbf{R} = \varpi_n^* \mathbf{1}'\mathbf{R}/n = \varpi_n^* \bar{R} \tag{35}$$

$$c = \omega^{*\prime}\Sigma\omega^* = \varpi_n^{*2}\kappa_1/n \tag{36}$$

$$\mu(dy) = \lambda\nu(dz) \text{ where } y = \omega^{*\prime}\mathbf{J}z = \varpi_n^* \bar{J} z \mathbf{1}'\mathbf{1}/n = \varpi_n^* \bar{J} z \tag{37}$$

with the second equation above following from $\Sigma = \bar{\Sigma} + \Sigma^\perp$ with $\mathbf{1}'\bar{\Sigma}\mathbf{1} = \kappa_1 \mathbf{1}'\mathbf{1}\mathbf{1}'\mathbf{1}/\mathbf{n} = \kappa_1 n$ and $\mathbf{1}'\Sigma^\perp\mathbf{1} = \mathbf{0}$.

All three of these quantities are $O(1)$ as $n \to \infty$, which means that the diversification effects are extremely weak. Moreover, since the Lévy measure is $O(1)$, the optimal portfolio in this case is not much better protected against contagion than a nondiversified portfolio.

B. *Different Expected Excess Returns*

The situation changes when we allow the n risky assets to have different expected excess returns while retaining the homogeneous covariance and jumps. Decomposing on the same basis as above, let

$$\mathbf{R} = \bar{R}\mathbf{1} + \mathbf{R}^\perp \tag{38}$$

with $\left\|\mathbf{R}^\perp\right\|^2 = \mathbf{R}^{\perp\prime}\mathbf{R}^\perp = O(n)$, while $\mathbf{J}^\perp = \mathbf{0}$, so that $\mathbf{J} = \bar{J}\mathbf{1}$. The optimal portfolio solution, as above, can be decomposed as $\omega = \bar{\omega}\mathbf{1} + \omega^\perp$ so that the minimization problem again separates as in (29), where we now have

$$g^\perp(\omega^\perp) = -\gamma \omega^{\perp\prime}\mathbf{R}^\perp + \frac{1}{2}\gamma^2 \omega^{\perp\prime}\kappa_2 \left(\mathbf{I} - \frac{1}{n}\mathbf{1}\mathbf{1}' \right) \omega^\perp \tag{39}$$

$$\bar{g}(\bar{\omega}) = -\gamma n \bar{\omega} \bar{R} + \frac{1}{2}\gamma^2 n^2 \bar{\omega}^2 \kappa_1 \frac{1}{n} + \lambda \int_0^1 \left(e^{-\gamma n \bar{\omega} \bar{J} z} - 1 \right) \nu(dz). \tag{40}$$

The first order condition for minimizing (39) is

$$-\gamma \mathbf{R}^{\perp} + \gamma^2 \kappa_2 \left(\mathbf{I} - \frac{1}{n} \mathbf{1}\mathbf{1}' \right) \omega^{\perp *} = 0$$

whose solution is

$$\omega^{\perp *} = \frac{1}{\gamma \kappa_2} \mathbf{R}^{\perp} = \frac{1}{\gamma v^2 (1 - \rho)} \mathbf{R}^{\perp} \tag{41}$$

since $\mathbf{1}'\omega^{\perp *} = 0$.

Since $\bar{g}$ in (40) is unaffected by the presence of $\mathbf{R}^{\perp}$, the optimal solution for $\bar{\omega}$ is identical to that given above, namely $\bar{\omega}^* = \varpi_n^*/n$ where ϖ_n^* is given in (33). Also, as before, in the limit where $n \to \infty$, we have again $\varpi_n^* \to \varpi_\infty^*$ where ϖ_∞^* is given in (34).

Therefore, the optimal discounted wealth process Y_t follows an arithmetic Lévy process with the characteristic triple

$$b = \varpi_n^* \bar{R} + \frac{1}{\gamma v^2 (1 - \rho)} \mathbf{R}^{\perp \prime} \mathbf{R}^{\perp} \tag{42}$$

$$c = \varpi_n^{*2} \kappa_1 / n + \frac{1}{\gamma^2 v^2 (1 - \rho)} \mathbf{R}^{\perp \prime} \mathbf{R}^{\perp} \tag{43}$$

$$\mu(dy) = \lambda \nu(dz) \quad \text{where} \quad y = \varpi_n^* \bar{J} z. \tag{44}$$

Here, since $\mathbf{R}^{\perp \prime}\mathbf{R}^{\perp} = O(n)$ and $\kappa_1 = O(n)$ while $\kappa_2 = O(1)$, we have that b and c are $O(n)$, due to the second term in equations (42) and (43) respectively, while the Lévy measure remains $O(1)$ as $n \to \infty$. This means that nonhomogeneous expected excess returns $\mathbf{R}^{\perp}$ lead the investor to place a linearly increasing amount of wealth in the risky assets as n grows, which in turns leads to increasing expected returns b and variance c, both growing linearly in the number of assets. On the other hand, as n grows, the exposure to contagion jumps remains bounded, and is dwarfed by the exposure to diffusive risk.

Indeed, the additional investment in the risky assets due to the presence of $\mathbf{R}^{\perp}$ is entirely in the direction of $\omega^{\perp}$, which is orthogonal to $\mathbf{J}$. So these additional amounts invested in the risky assets are all achieved with zero net additional exposure to the jump risk. Thus, compared to the case where $\mathbf{R}^{\perp} = \mathbf{0}$, the presence of $\mathbf{R}^{\perp}$ with $\mathbf{R}^{\perp \prime}\mathbf{R}^{\perp} = O(n)$ allows the CARA investor to optimize in such a way as to increase their expected gains (at the expense of increased variance resulting from the diffusive risk), while keeping the exposure to jumps fixed.

C. Examples of Jump Distributions

To fully compute the optimal policy, we need to specify the distribution $\nu(dz)$ driving the common jumps; then we can compute the integral in (34). As a first example, consider the special case where Z is distributed on $[0,1]$ according to a uniform distribution, $\nu(dz) = dz$. As a second example, consider the case where

$$\nu(dz) = \frac{\alpha^2(1-z)\exp(-\alpha z)}{\alpha - 1 + \exp(-\alpha)} dz \tag{45}$$

for $\alpha > 0$. The density function is monotonically decreasing from 0 to 1 (where it reaches a value of 0). The mean of that distribution is $(\alpha - 2 + (\alpha+2)\,e^{-\alpha})\,/((\alpha - 1 + e^{-\alpha})\,\alpha)$. The higher α, the closer the mean to 0. When $\alpha \to 0$, the mean goes to 1/3.

Equation (34) specializes to

$$\varpi_\infty^* = \arg\min_{\{\varpi\}}\ f_\infty(\varpi) \tag{46}$$

where using a uniform distribution

$$\begin{aligned} f_\infty(\varpi) &= -\gamma\varpi\bar{R} + \frac{1}{2}\gamma^2\varpi^2 v^2\rho + \lambda\left[\int_0^1 e^{-\gamma\varpi\bar{J}z}dz - 1\right] \\ &= -\gamma\varpi\bar{R} + \frac{1}{2}\gamma^2\varpi^2 v^2\rho + \lambda\left[\frac{e^{-\gamma\varpi\bar{J}} - 1}{-\gamma\varpi\bar{J}} - 1\right]. \end{aligned} \tag{47}$$

Figure 1 plots the objective function, $f_n(\varpi) = \bar{g}(\varpi/n)$ in the special case where R is constant and Z is uniformly distributed on $[0,1]$, and shows its convergence to $f_\infty(\varpi)$ as $n \to \infty$, along with $\arg\min f_n(\varpi) = \varpi_n^*$, converging to ϖ_∞^*.

D. Response to Jumps of Different Arrival Intensity

The following comparative statics are of interest. We have

$$\varpi_\infty^* \to -\infty \text{ as } \lambda \to \infty \tag{48}$$

$$\varpi_\infty^* \to \frac{\bar{R}}{\gamma v^2 \rho} \text{ as } \lambda \to 0. \tag{49}$$

The first limit means that the investor will go short to an unbounded extent on all the risky assets if the arrival rate of the jumps goes to infinity. This is to be expected, since

$\bar{J} < 0$. If the jumps become less and less frequent, then ϖ_∞^* tends to a finite limit driven by the diffusive characteristics of the assets. In particular, the higher the variance of the assets and/or the more heavily correlated they are, the smaller the investment in each one of them. The higher the expected excess return of the assets ($\bar{R}$) relative to the investor's risk aversion (γ), the higher the amount invested.

E. *Jumps vs. Expected Return Trade-off*

If $\bar{R} > 0$, there exists a critical value $\tilde{\lambda}$ such that

$$\varpi_\infty^* > 0 \text{ for } \lambda < \tilde{\lambda} \tag{50}$$

$$\varpi_\infty^* \leq 0 \text{ for } \lambda \geq \tilde{\lambda}. \tag{51}$$

That is, as long as jumps do not occur too frequently ($\lambda < \tilde{\lambda}$), the investor will go long on the assets in order to capture their expected return, even though that involves taking on the (negative) risk of the jumps. When the jumps occur frequently enough ($\lambda \geq \tilde{\lambda}$), then the investor decides to forgo the expected return of the assets and focuses on canceling his exposure to the jump risk by going short these assets.

F. *Flight to Quality*

Starting from a situation where $\lambda < \tilde{\lambda}$, if the perception of the jump risk increases ($\lambda \uparrow \tilde{\lambda}$), then the optimal policy is for the investor to flee-to-quality, by dumping the risky assets indiscriminately ($\varpi_\infty^* \downarrow 0$) and investing the proceeds in the riskless asset. Because we are not imposing short sale constraints, if the perception of the jump risk exceeds $\tilde{\lambda}$, then the investor should go even further and start short–selling the risky assets.

The critical value $\tilde{\lambda}$ is given by

$$\tilde{\lambda} = -\frac{\bar{R}}{\bar{J} \int_0^1 z\nu\,(dz)} \tag{52}$$

Clearly, the higher $\bar{R}$ relative to $(-\bar{J})$, the higher $\tilde{\lambda}$. And the smaller the expected value of Z, the bigger $\tilde{\lambda}$. Using a uniform $\nu\,(dz)$, we get

$$\tilde{\lambda} = -2\frac{\bar{R}}{\bar{J}}. \tag{53}$$

Now, if $\bar{R} \leq 0$, then $\varpi_\infty^* \leq 0$ for every $\lambda \geq 0$. In that case, there is no point in ever going long those assets since both the expected return and the jump components negatively impact the investor's rate of return.

In the second example, we obtain

$$\tilde{\lambda} = -\frac{\bar{R}\,(\alpha - 1 + e^{-\alpha})\,\alpha}{\bar{J}\,(\alpha - 2 + (\alpha + 2)\,e^{-\alpha})}. \tag{54}$$

IV. Optimal Portfolio in the Presence of Jumps in a Multi–Sector Economy

We now study the selection problem in an economy composed of m sectors (or regions of the world), each containing k firms (or countries). The total number of assets available to the investor is $n = mk$. We are interested in the situation where m is fixed and k goes to infinity with n.

A. *No Cross–Sectorial Diffusive Correlation*

We start with the situation where the diffusive risk generates correlated returns within sectors, but not across sectors. The only source of cross-sectorial correlation is due to the jumps. In that case, the sectorial decomposition we assume corresponds to the following block form of the covariance matrix Σ due to the diffusive part:

$$\underset{n\times n}{\Sigma} = \begin{pmatrix} \Sigma_1 & 0 & \cdots \\ 0 & \ddots & 0 \\ \cdots & 0 & \Sigma_m \end{pmatrix} \tag{55}$$

is a block diagonal matrix with blocks

$$\underset{k\times k}{\Sigma_l} = v_l^2 \begin{pmatrix} 1 & \rho_l & \cdots \\ \rho_l & \ddots & \rho_l \\ \cdots & \rho_l & 1 \end{pmatrix} \tag{56}$$

where $v_l^2 > 0$ is the variance of returns due to the diffusive shocks in the $l-$sector and $1 > \rho_l > 0$ is their correlation.

The spectral decomposition of the Σ matrix is

$$\Sigma \quad = \quad \sum_{l=1}^{m} \kappa_{1l} \frac{1}{k} \mathbf{1}_l \mathbf{1}_l' \quad + \quad \sum_{l=1}^{m} \kappa_{2l} \left(\mathbf{F}_l - \frac{1}{k} \mathbf{1}_l \mathbf{1}_l' \right) \quad = \quad \bar{\Sigma} + \Sigma^{\perp} \tag{57}$$

where

$$\kappa_{1l} = v_l^2 + v_l^2 (k-1) \rho_l \tag{58}$$

$$\kappa_{2l} = v_l^2 (1 - \rho_l) \tag{59}$$

are the $2m$ distinct eigenvalues of Σ. The multiplicity of each κ_{1l} is 1, and the multiplicity of each κ_{2l} is $k-1$. The eigenvector for κ_{1l} is $\mathbf{1}_l$, the $n-$vector with ones placed in the rows corresponding to the $l-$block and zeros everywhere else, that is

$$\mathbf{1}_l = [0, \ldots, 0, \underbrace{1, \ldots, 1}_{\text{sector } l}, 0, \ldots, 0]', \tag{60}$$

where the first 1 is located in the $k(l-1)+1$ coordinate. $\mathbf{F}_l$ is an $n \times n$ block diagonal matrix with a $k \times k$ identity matrix $\mathbf{I}_k$ placed in the $l-$block and zeros everywhere else:

$$\underset{n \times n}{\mathbf{F}_l} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \mathbf{I}_k & \vdots \\ 0 & \cdots & 0 \end{pmatrix}, \tag{61}$$

Corresponding to the above spectral structure, we have the orthogonal decomposition $\mathbb{R}^n = \bar{V} \oplus V^{\perp}$ where $\bar{V}$ is the span of $\{\mathbf{1}_l\}_{l=1,..,m}$.

As for the jump vector $\mathbf{J}$ in our m–sector economy, we assume that $\mathbf{J} \in \bar{V}$:

$$\mathbf{J} = \sum_{l=1}^{m} j_l \mathbf{1}_l = [\underbrace{j_1, \ldots, j_1}_{\text{sector 1}}, \underbrace{j_2, \ldots, j_2}_{\text{sector 2}}, \cdots, \underbrace{j_m, \ldots, j_m}_{\text{sector } m}]' \tag{62}$$

meaning that firms within a given sector have the same response to the arrival of a Poisson jump, i.e., to a change in N_t. But the proportional response of firms of different sectors to the arrival of a jump can be different.

Finally, we assume that the vector of expected excess returns has the form

$$\mathbf{R} = \sum_{l=1}^{m} r_l \mathbf{1}_l + \mathbf{R}^{\perp} = \bar{\mathbf{R}} + \mathbf{R}^{\perp}. \tag{63}$$

Here, we allow the expected excess returns to differ both within and across sectors, by allowing $\mathbf{R}^{\perp} \neq \mathbf{0}$. The general $\mathbf{R}^{\perp}$ is orthogonal to each $\mathbf{1}_l$ and has the form

$$\mathbf{R}^{\perp} = [\mathbf{R}_1^{\perp\prime}, ..., \mathbf{R}_m^{\perp\prime}]'$$

where each of the k-vectors $\mathbf{R}_l^{\perp}$ is orthogonal to the k-vector $\mathbf{1}$. As in section B, we may suppose that each component of $\mathbf{R}^{\perp}$ is $O(1)$.

With this structure, we will be looking for a vector of optimal portfolio weights of the form

$$\mathbf{\Omega} \ = \ \sum_{l=1}^{m} \omega_l \mathbf{1}_l \, + \omega^{\perp} = \bar{\omega} + \omega^{\perp}. \tag{64}$$

The minimization problem again separates as

$$\left(\omega^{\perp *}, \bar{\omega}^{*}\right) = \arg \min_{\left\{\omega^{\perp}, \bar{\omega}\right\}} \left\{g^{\perp}(\omega^{\perp}) + \bar{g}(\bar{\omega})\right\} \tag{65}$$

where

$$g^{\perp}(\omega^{\perp}) = -\gamma \omega^{\perp\prime} \mathbf{R}^{\perp} + \frac{1}{2}\gamma^2 \omega^{\perp\prime} \Sigma^{\perp} \omega^{\perp} \tag{66}$$

$$\bar{g}(\bar{\omega}) = -\gamma k \sum_{l=1}^{m} \omega_l r_l + \frac{1}{2}\gamma^2 k \sum_{l=1}^{m} \omega_l^2 \kappa_{1l} + \lambda \int_0^1 \left[e^{-\gamma z k \sum_{l=1}^{m} \omega_l j_l} - 1\right] \nu\left(dz\right). \tag{67}$$

The first order condition for minimizing $g^{\perp}(\omega^{\perp})$ leads to the optimal solution

$$\omega^{\perp *} = \frac{1}{\gamma} \Sigma^{-1} \mathbf{R}^{\perp} \tag{68}$$

where, by the block diagonal form of Σ in (55), can be written in the form $\omega^{\perp *} = [\omega_1^{\perp *\prime}, ..., \omega_m^{\perp *\prime}]'$ with

$$\omega_l^{\perp *} = \frac{1}{\gamma \kappa_{2l}} \mathbf{R}_l^{\perp}$$

for $l = 1, \ldots, m$.

The problem of minimizing $\bar{g}(\bar{\omega})$ is an analogue of what happens with one sector, as in Section III, but in dimension m, and similarly its solution has a limit as k goes to infinity with n (the number of sectors m being fixed). With the change of variable $\varpi_n = k\bar{\omega}$ we see that

$$\varpi_n^{*} = \arg \min_{\{\varpi_n\}} \left\{ -\gamma \sum_{l=1}^{m} \varpi_{ln} r_l + \frac{1}{2}\gamma^2 \sum_{l=1}^{m} \varpi_{ln}^2 \kappa_{1l}/k + \lambda \int_0^1 \left[e^{-\gamma z \sum_{l=1}^{m} \varpi_{ln} j_l} - 1\right] \nu\left(dz\right) \right\} \tag{69}$$

Letting $k \to \infty$, we get $\varpi_n^* \to \varpi_\infty^*$ where

$$\varpi_\infty^* = \arg\min_{\{\varpi_\infty\}} \left\{ -\gamma \sum_{l=1}^m \varpi_{l\infty} r_l + \frac{1}{2}\gamma^2 \sum_{l=1}^m \varpi_{l\infty}^2 v_l^2 \rho_l + \lambda \int_0^1 \left[e^{-\gamma z \sum_{l=1}^m \varpi_{l\infty} j_l} - 1 \right] \nu(dz) \right\}, \tag{70}$$

which, compared to (34), is an $m-$dimensional minimization problem, instead of a one-dimensional one. But the convexity of the objective function implies the existence of the minimizer.

The discounted wealth process Y_t^* of the optimizing investor will thus have arithmetic Lévy dynamics with ω_t given by $\bar{\omega}^* + \omega^{\perp *}$. The characteristic triple of Y_t^* is then

$$b = \sum_{l=1}^m \varpi_{ln}^* r_l + \sum_{l=1}^m \frac{1}{\gamma v_l^2 (1-\rho_l)} \mathbf{R}_l^{\perp\prime} \mathbf{R}_l^{\perp} \tag{71}$$

$$c = \sum_{l=1}^m \varpi_{ln}^{*2} \kappa_{1l}/k + \sum_{l=1}^m \frac{1}{\gamma^2 v_l^2 (1-\rho_l)} \mathbf{R}_l^{\perp\prime} \mathbf{R}_l^{\perp} \tag{72}$$

$$\mu(dy) = \lambda \nu(dz) \quad \text{where} \quad y = \sum_{l=1}^m \varpi_{ln}^* j_l z. \tag{73}$$

Under the natural condition that $\mathbf{R}_l^{\perp\prime} \mathbf{R}_l^{\perp} = O(k)$, we are lead to conclusions similar to those we drew in section B. As k, the number of stocks per sector, increases, the optimal portfolio can achieve expected gains at the expense of variance which both grow approximately linearly with k, while keeping the exposure to jumps bounded. Essentially this result is achieved by the investor apportioning an increasing fraction of assets in the subspace orthogonal to the vectors $\mathbf{1}_l$.

For the two examples of $\nu(dz)$ of section C we get the following objective functions in the two-sector case:

$$\begin{aligned} f_n(\varpi) = & -\gamma (\varpi_1 r_1 + \varpi_1 r_2) + \frac{1}{2}\gamma^2 \left(\varpi_1^2 v_1^2 \frac{\kappa_1}{k} + \varpi_2^2 v_2^2 \frac{\kappa_2}{k} \right) \\ & + \lambda \left[\frac{e^{-\gamma\left(\varpi_1 \bar{J}_1 + \varpi_2 \bar{J}_2\right)} - 1}{-\gamma \left(\varpi_1 \bar{J}_1 + \varpi_2 \bar{J}_2 \right)} - 1 \right], \end{aligned} \tag{74}$$

and

$$
\begin{aligned}
f_n(\varpi) = & -\gamma(\varpi_1 r_1 + \varpi_1 r_2) + \frac{1}{2}\gamma^2\left(\varpi_1^2 v_1^2 \frac{\kappa_1}{k} + \varpi_2^2 v_2^2 \frac{\kappa_2}{k}\right) \\
& + \lambda\left(\frac{\alpha^2}{\alpha - 1 + \exp(-\alpha)}\right) \\
& \times \left[\left(\frac{\alpha + \gamma\left(\varpi_1 \bar{J}_1 + \varpi_2 \bar{J}_2\right) - 1 + \exp(-\left(\alpha + \gamma\left(\varpi_1 \bar{J}_1 + \varpi_2 \bar{J}_2\right)\right))}{\left(\alpha + \gamma\left(\varpi_1 \bar{J}_1 + \varpi_2 \bar{J}_2\right)\right)^2}\right) - 1\right]
\end{aligned}
\tag{75}
$$

respectively. Figure 2 plots the objective function, $f_n(\varpi) = \bar{g}(\varpi/k)$ that we obtain in a two-sector economy in the special case where Z is uniformly distributed on $[0,1]$, that is the function (74).

B. *Cross–Sectorial Diffusive Correlation*

We now allow the diffusive risk to generate correlated returns within sectors, as well as across sectors. In addition, the jumps generate cross-sectorial correlation. That is, we assume that

$$
\underset{n\times n}{\Sigma} = \begin{pmatrix} \Sigma_1 & v^2\rho_0 & \cdots \\ v^2\rho_0 & \ddots & v^2\rho_0 \\ \cdots & v^2\rho_0 & \Sigma_m \end{pmatrix}
$$

is a block diagonal matrix with blocks:

$$
\underset{k\times k}{\Sigma_l} = v^2 \begin{pmatrix} 1 & \rho & \cdots \\ \rho & \ddots & \rho \\ \cdots & \rho & 1 \end{pmatrix}
$$

where $1 > \rho > \rho_0 \geq 0$ and, as before, $n = mk$. For simplicity, we make the within-sector correlation coefficient ρ identical across sectors. This Σ matrix has three distinct eigenvalues

$$
\begin{aligned}
\kappa_1 &= v^2 + v^2 k(m-1)\rho_0 + v^2(k-1)\rho \\
\kappa_2 &= v^2 - v^2 k\rho_0 + v^2(k-1)\rho \\
\kappa_3 &= v^2(1-\rho)
\end{aligned}
$$

with multiplicity 1, $m-1$ and $(k-1)m$ respectively. In parallel with the previous section, we focus on the orthogonal decomposition $\mathbb{R}^n = \bar{V} \oplus V^{\perp}$ where $V^{\perp}$, the κ_3-eigenspace, consists of vectors orthogonal to each $\mathbf{1}_l$ and $\bar{V}$ is the m-dimensional subspace spanned by the vectors $\{\mathbf{1}_l\}_{l=1,...,m}$.

We again assume that $\mathbf{J} = \sum_{l=1}^{m} j_l \mathbf{1}_l \in \bar{V}$, in other words, that firms within the same sector have the same response to the arrival of a Poisson jump. The vector of expected excess returns again has the form

$$\mathbf{R} = \sum_{l=1}^{m} r_l \mathbf{1}_l + \mathbf{R}^{\perp} = \bar{\mathbf{R}} + \mathbf{R}^{\perp}. \tag{76}$$

where the general $\mathbf{R}^{\perp}$ is orthogonal to each $\mathbf{1}_l$ and has the form

$$\mathbf{R}^{\perp} = [\mathbf{R}_1^{\perp\prime}, ..., \mathbf{R}_m^{\perp\prime}]'$$

As in section B, we may suppose that each component of $\mathbf{R}^{\perp}$ is $O(1)$.

The vector of optimal portfolio weights has the form

$$\mathbf{\Omega} \ = \ \sum_{l=1}^{m} \omega_l \mathbf{1}_l + \omega^{\perp} = \bar{\omega} + \omega^{\perp}. \tag{77}$$

The minimization problem again separates as

$$\left(\omega^{\perp *}, \bar{\omega}^*\right) = \arg \min_{\{\omega^{\perp}, \bar{\omega}\}} \left\{ g^{\perp}(\omega^{\perp}) + \bar{g}(\bar{\omega}) \right\} \tag{78}$$

where now

$$g^{\perp}(\omega^{\perp}) = -\gamma \omega'^{\perp} \mathbf{R}^{\perp} + \frac{\kappa_3 \gamma^2}{2} \omega^{\perp\prime} \omega^{\perp} \tag{79}$$

$$\begin{aligned} \bar{g}(\bar{\omega}) = & -\gamma k \sum_{l=1}^{m} \omega_l r_l + \frac{\gamma^2}{2} \frac{k\kappa_1}{m} \left(\sum_{l=1}^{m} \omega_l \right)^2 \\ & + \frac{\gamma^2}{2} k\kappa_2 \left[\sum_{l=1}^{m} \omega_l^2 - \frac{1}{m} \left(\sum_{l=1}^{m} \omega_l \right)^2 \right] \\ & + \lambda \int_0^1 \left[e^{-\gamma z k \sum_{l=1}^{m} \omega_l j_l} - 1 \right] \nu\left(dz\right). \end{aligned} \tag{80}$$

The essential structure of the solution is similar to that of the previous section. First, the minimizer of $g^{\perp}(\omega^{\perp})$ can be written in the form $\omega^{\perp *} = [\omega_1^{\perp *\prime}, ..., \omega_m^{\perp *\prime}]'$ with

$$\omega_l^{\perp *} = \frac{1}{\gamma \kappa_3} \mathbf{R}_l^{\perp}$$

for $l = 1, \ldots, m$. The m-dimensional minimization of $\bar{g}(\bar{\omega})$ can always be solved, and the resulting solution has n dependence similar to before. With the change of variable $\varpi_n = k\bar{\omega}$ we see that the minimizer $\varpi_n^* = \arg\min_{\{\varpi_n\}} \bar{g}(\varpi_n/k)$ will have the limit $\varpi_n^* \to \varpi_\infty^*$ as $k \to \infty$, where

$$\begin{aligned}\varpi_\infty^* = \arg\min_{\{\varpi_\infty\}} \Bigg\{ -\gamma \sum_{l=1}^m \varpi_{l\infty} r_l + \frac{\gamma^2 v^2}{2} \Bigg\{ & \frac{(m-1)\rho_0 + \rho}{m} \left(\sum_{l=1}^m \varpi_{l\infty} \right)^2 \\ & + (\rho - \rho_0) \left[\sum_{l=1}^m \varpi_{l\infty}^2 - \frac{1}{m} \left(\sum_{l=1}^m \varpi_{l\infty} \right)^2 \right] \Bigg\} \\ & + \lambda \int_0^1 \left[e^{-\gamma z \sum_{l=1}^m \varpi_{l\infty} j_l} - 1 \right] \nu(dz) \Bigg\},\end{aligned} \tag{81}$$

The discounted wealth process Y_t^* of the optimizing investor will thus have arithmetic Lévy dynamics with ω_t given by $\bar{\omega}^* + \omega^{\perp *}$. The characteristic triple of Y_t^* is then

$$b = \sum_{l=1}^m \varpi_{ln}^* r_l + \frac{1}{\gamma v^2 (1-\rho)} \sum_{l=1}^m \mathbf{R}_l^{\perp\prime} \mathbf{R}_l^{\perp} \tag{82}$$

$$c = \frac{\kappa_1 - \kappa_2}{n} \left(\sum_{l=1}^m \varpi_{ln}^* \right)^2 + \frac{\kappa_2}{k} \sum_{l=1}^m \varpi_{ln}^{*2} + \frac{1}{\gamma^2 v^2 (1-\rho)} \sum_{l=1}^m \mathbf{R}_l^{\perp\prime} \mathbf{R}_l^{\perp} \tag{83}$$

$$\mu(dy) = \lambda \nu(dz) \quad \text{where} \quad y = \sum_{l=1}^m \varpi_{ln}^* j_l z. \tag{84}$$

Under the natural condition that $\mathbf{R}_l^{\perp\prime} \mathbf{R}_l^{\perp} = O(k)$ the conclusions are the same as those of the previous section: As k, the number of stocks per sector, increases, the optimal portfolio can achieve expected gains at the expense of variance which both grow approximately linearly with k, while keeping the exposure to jumps bounded.

V. Partial Response to the Jumps

Suppose now that the true model is as before but the investor (mistakenly) thinks instead that the risky assets are driven by Brownian motions only, with no jumps:

$$\frac{dS_{i,t}}{S_{i,t-}} = \left(r + \hat{R}_i \right) dt + \sum_{j=1}^n \hat{\sigma}_{i,j} dW_{j,t}, \ i = 1, \ldots, n. \tag{85}$$

While this investor ignores the Poisson process and assumes that returns are driven only by Brownian motions, he still accounts correctly for the additional variance and covariances generated by the jumps.

A. A Misspecified Model that Matches the First Two Moments

In this case the total expected excess returns are

$$\hat{R}_i dt = E_t \left[\frac{dS_{i,t}}{S_{i,t-}} \right] - r dt = \left(R_i + J_i \lambda \bar{Z} \right) dt \tag{86}$$

and the total variance-covariance matrix $\hat{\Sigma} = \hat{\sigma}\hat{\sigma}'$ is given by

$$\hat{\Sigma} dt = E_t \left[\left(\frac{d\mathbf{S}_t}{\mathbf{S}_{t-}} \right) \left(\frac{d\mathbf{S}_t}{\mathbf{S}_{t-}} \right)' \right] = \left(\Sigma + \lambda \mathbf{J}\mathbf{J}' \bar{Z}^2 \right) dt \tag{87}$$

Now the investor thinks that ω_t^{NJ*} has to satisfy:

$$\omega^{NJ*} = \arg \min_{\{\omega^{NJ}\}} g(\omega^{NJ}) \tag{88}$$

where

$$g(\omega^{NJ}) = -\gamma \left(\omega^{NJ} \right)' \hat{\mathbf{R}} + \frac{1}{2} \gamma^2 \left(\omega^{NJ} \right)' \hat{\Sigma} \omega^{NJ} \tag{89}$$

We again assume a one sector economy with different expected excess returns, then

$$\begin{aligned} \hat{\mathbf{R}} &= \left(\bar{R} + \lambda \bar{J} \bar{Z} \right) \mathbf{1} + \mathbf{R}^{\perp}, \\ \hat{\Sigma} &= \left(\bar{\Sigma} + \lambda \bar{J}^2 \bar{Z}^2 \right) \mathbf{1}\mathbf{1}' + \Sigma^{\perp}, \\ \omega^{NJ} &= \bar{\omega}^{NJ} \mathbf{1} + \omega^{\perp}, \end{aligned}$$

thus, the minimization problem becomes

$$\left(\omega^{\perp *}, \bar{\omega}^{NJ*} \right) = \arg \min_{\left\{ \omega^{\perp}, \bar{\omega}^{NJ} \right\}} \left\{ g^{\perp}(\omega^{\perp}) + \bar{g}(\bar{\omega}^{NJ}) \right\} \tag{90}$$

where

$$g^{\perp}(\omega^{\perp}) = -\gamma \omega^{\perp\prime} \mathbf{R}^{\perp} + \frac{1}{2} \gamma^2 \omega^{\perp\prime} \Sigma^{\perp} \omega^{\perp}, \tag{91}$$

$$\bar{g}(\bar{\omega}^{NJ}) = -\gamma n \bar{\omega}^{NJ} \left(\bar{R} + \lambda \bar{J} \bar{Z} \right) + \frac{1}{2} \gamma^2 n^2 \left(\bar{\omega}^{NJ} \right)^2 \left(\bar{\Sigma} + \lambda \bar{J}^2 \bar{Z}^2 \right) \tag{92}$$

The optimal solution for $\omega^{\perp}$ in this case is

$$\omega^{\perp *} = \frac{1}{\gamma v^2 (1-\rho)} \mathbf{R}^{\perp}. \tag{93}$$

As for the optimal solution for $\bar{\omega}^{NJ}$, with the change of variable $\varpi_n^{NJ} = n\bar{\omega}^{NJ}$, we see

$$\varpi_n^{NJ*} = \frac{1}{\gamma} \left(\frac{\kappa_1}{n} + \lambda \bar{J}^2 \bar{Z}^2 \right)^{-1} \left(\bar{R} + \lambda \bar{J} \bar{Z} \right). \tag{94}$$

Letting $n \to \infty$, we have that $\varpi_n^{NJ*} \to \varpi_\infty^{NJ*}$ where

$$\varpi_\infty^{NJ*} = \frac{1}{\gamma} \left(v^2 \rho + \lambda \bar{J}^2 \bar{Z}^2 \right)^{-1} \left(\bar{R} + \lambda \bar{J} \bar{Z} \right). \tag{95}$$

It is important to notice that $\omega^{\perp *}$ is the same whether the investor recognizes the presence of jumps or not.

B. *Higher Moment Effect*

On the other hand, matching the first two moments is not sufficient to fully deliver the optimal solution. We can compare ϖ_∞^{NJ*} to

$$\varpi_\infty^* = \arg \min_{\{\varpi_\infty\}} \left\{ -\gamma \varpi_\infty \bar{R} + \frac{1}{2} \gamma^2 \varpi_\infty^2 v^2 \rho + \lambda \int_0^1 \left[e^{-\gamma \varpi_\infty \bar{J} z} - 1 \right] \nu(dz) \right\}$$

Taking the third order Taylor's expansion of $\int_0^1 \left[e^{-\gamma \varpi_\infty \bar{J} z} - 1 \right] \nu(dz)$ with respect to ϖ_∞, we get

$$\begin{aligned} \int_0^1 \left[e^{-\gamma \varpi_\infty \bar{J} z} - 1 \right] \nu(dz) = & -\gamma \bar{J} \varpi_\infty \bar{Z} + \frac{1}{2} \gamma^2 \bar{J}^2 \varpi_\infty^2 \bar{Z}^2 \\ & - \frac{1}{3!} \gamma^3 \bar{J}^3 \varpi_\infty^3 \bar{Z}^3 + o\left(\varpi_\infty^3\right), \end{aligned} \tag{96}$$

then

$$\begin{aligned} \varpi_\infty^* = \arg \min_{\{\varpi_\infty\}} \Big\{ & -\gamma \varpi_\infty \left(\bar{R} + \lambda \bar{J} \bar{Z} \right) + \frac{1}{2} \gamma^2 \varpi_\infty^2 \left(\bar{\Sigma} + \lambda \bar{J}^2 \bar{Z}^2 \right) \\ & - \frac{1}{3!} \gamma^3 \bar{J}^3 \varpi_\infty^3 \bar{Z}^3 + o\left(\varpi_\infty^3\right) \Big\}. \end{aligned}$$

where $\bar{Z}^3 = \int_0^1 z^3 \nu(dz)$. Hence, if $\bar{J} < 0$ then we have

$$\varpi_\infty^* < \varpi_\infty^{NJ*}.$$

This result suggests that the investor who recognizes the presence of jumps will invest more money in the riskless asset. This is an effect driven by differences in the higher moments of the two processes, the one that is correctly specified (with jumps) and the one that is misspecified (no jumps but still a partial adjustment consisting of matching the first two moments correctly.)

VI. Example: Worldwide Asset Allocation

In this section, we compute the investor's optimal portfolio for a three-sector economy, representing three regions of the world. This example is not meant to be fully realistic, bur merely to illustrate the method. Each region contains four countries, for a total of twelve risky assets, each correponding to a country-wide equity index. The United States (USA), Germany (GE), Japan (JP) and the United Kingdom (UK) belong to the first region. The second region consists of Mexico (MX), Argentina (ARG), Brazil (BR) and Chile (CL). The countries in the third region are China (CH), Hong Kong (HK), Singapore (SNP) and Thailand (THA).

These equity indexes together with a riskless asset are the available investment opportunities for an investor. They are valued in U.S. dollars and follow an exponential Lévy process of the form (2). We assume that Z is distributed on $[0, 1]$ according to a uniform distribution. We allow the diffusive risk to generate correlated returns within regions as well as across regions, using the model of Section B. Table I reports the parameter values that we assume for the returns process in order to compute the investor's optimal portfolio. The values of the other parameters are $r = 0.06$ and $\gamma = 2$. The negative values of $\mathbf{J}$ reflect the empirical fact that the skewness of returns for all regions is negative, and for the emerging-markets it is more strongly negative.

It is then straightforward to compute the investor's optimal portfolio, ω^*, by applying the methodology described above. In addition, we calculate the optimal portfolio ω^{NJ*} of an investor who partially ignores jumps and assumes returns are given by a Gaussian process with the first two moments calibrated to those of the correctly specified model. The elements of the vector $\omega^{\perp *}$ are plotted in Figure 3. As expected, $\omega^{\perp *}$ is not affected by the jump parameters. What is more, $\omega^{\perp *}$ captures only diversification within regions.

Panels A, B, and C in Figure 4 show $\bar{\omega}^* = (\bar{\omega}_1^*, \bar{\omega}_2^*, \bar{\omega}_3^*)$ and $\bar{\omega}^{NJ*} = \left(\bar{\omega}^{NJ*}, \bar{\omega}^{NJ*}, \bar{\omega}^{NJ*}\right)$ for values of the jump intensity parameter $\lambda = 1$, $\lambda = 1/5$ and $\lambda = 1/10$. Since those are annualized values, $\lambda = 1/10$ means that on average the probability of getting one jump in any given year is 10%, or a jump is expected to occur once every 10 years. We calculate $\bar{\omega}^*$ by minimizing $\bar{g}(\bar{\omega})$, and $\bar{\omega}^{NJ*}$ is computed directly as in (80). We observe that the effect of jumps on the optimal portfolio is stronger for an investor who accounts for jumps. Moreover, while the elements of $\bar{\omega}^{NJ*}$ converge to finite numbers, $\bar{\omega}_2^*$ and $\bar{\omega}_3^*$ decrease without limit as λ increases.

Panels D, E and F in Figure 4 plot the components of the portfolios weights ω^* and ω^{NJ*}. The plots suggest that diversification across regions is strongly influenced by the jump risk. As a result, for $\lambda = 1/5$ and $\lambda = 1$ the investor prefers to invest more heavily in Region 1 than in Regions 2 and 3. On the other hand, diversification within regions is mostly driven by the Brownian risk. Furthermore, when λ is small, the Brownian risk is the factor that plays the main diversification role. Finally, for every λ the amount invested in the riskless asset is smaller for an investor who partially ignores jumps and simply adjusts the first two moments of returns.

VII. Extensions, Limitations and Conclusions

We have proposed a new approach to characterize in closed form, up to a constant, the portfolio selection problem for an investor concerned with the possibility of market contagion effects, and who seeks to control this risk by diversification or other means. The framework is illustrated by certain families of asset return models of increasing complexity, where each family allows models for an unbounded number of assets. We can address certain questions. How exactly does increasing the number of available assets improve the investor's exposure to both diffusive and contagion risk? How does the portfolio of an investor who fears contagion differ from the portfolio of one who does not? Is there a simple form for the optimal portfolio which is achieved asymptotically as the number of assets grows to infinity? Is there a systematic way to add complexity to the market model while retaining computational tractability?

In this paper, the standard multi–asset geometric Brownian motion models is extended to an exponential Lévy models by the inclusion of contagion effects through a one dimensional jump distribution. For the general exponential Lévy model, the portfolio selection problem for n assets reduces to the minimization of a convex function in n dimensions. However, the crucial assumption of a special relation between the diffusive correlation matrix and the jump distribution, $\mathbf{J}^\perp = 0$, enables a further reduction of the problem to a convex optimization in the dimension of the number of sectors m, which we can take to be small while the total number of assets n is typically large. As an empirical strategy, one could imagine determining the number of sectors through spectral analysis or similar techniques, in order to determine endogenously the shape of the Σ matrix.

Our analysis allows us to draw the following three conclusions. First, our examples show that when the asset returns are sufficiently nonhomogeneous the total amount in the optimal portfolio invested in risky assets, hence the expected return and volatility, all grow linearly with n, while the exposure to the jumps remains bounded. Moreover, the optimal portfolio is asymptotically normally distributed as n gets large. Finally, the investor who correctly accounts for jumps always invests less in the risky assets than the investor who fails to include these jumps.

One can ask if these conclusions continue to hold and the approach remains valid when extensions and generalizations of this work are considered:

1. The first extension to consider is to allow the jump dimension to grow to the number of sectors m, while retaining the condition $\mathbf{J}^\perp = 0$. In that case, the selection problem again reduces to an m dimensional optimization, and hence that our main conclusions remain intact.

2. Another extension to consider is to generalize the cross sectorial correlation structure: it is clear that much more general correlation structures will preserve the condition $\mathbf{J}^\perp = 0$ and hence the dimensional reduction, leading to similar conclusions. In fact, one could consider modelling the spectral decomposition of the Σ matrix directly, instead of parametrizing the matrix itself and then determining its spectral decomposition. Of course, the form of the matrix is easier to interpret or derive from an economic model than its spectral decomposition, which argues for the indirect (or structural) approach. But the reduced form approach has the advantage of greater

generality, since we are no longer constrained to being able to derive the spectral decomposition explicitly from the assumed form of the Σ matrix.

3. A third extension is the situation of an investor with power utility (see Kallsen (2000), Choulli and Hurd (2001) in the context of jumps) can also be addressed straightforwardly in our framework, leading to the same separability into two separate problems for the portfolio weights. The main difference is that the power utility investor will place a constant fraction of wealth in each risky asset, rather than a constant amount. Apart from this change, the computations and hence the conclusions we can draw are exactly similar to the case of the exponential utility investor.

4. Stochastic volatility of the type considered in Liu, Longstaff, and Pan (2003) requires solving our nonlinear equations for weight vectors stepwise in time, in parallel with ordinary differential equations (which themselves depend on the current portfolio weights). This does not appear doable in closed form.

5. Portfolio constraints such as short-selling constraints are sometimes introduced into portfolio theory but, when generic constraints are imposed on the optimal portfolio, we cannot expect the dimensional reduction to be preserved or our conclusions to hold. However, a utility function such as power utility which becomes $-\infty$ for wealth below a finite threshold, sometimes automatically implies certain constraints: it appears that in this case, much of our analysis remains intact.

6. The final extension to consider is to allow $\mathbf{J}^{\perp} \neq 0$. In this case, the reduction in dimension breaks down: One is left with a problem of full complexity, and there is little of interest we can prove. While the condition $\mathbf{J}^{\perp} = 0$ is without economic justification, breaking it seems to be the last thing a mathematician would want to do because the extra generality does not justify the additional computational complexity of the solution. Since in any real life application there can never be enough market data to calibrate the jumps, the specification of the jumps will always be largely subjective. The main purpose of adding contagion jumps must be to stress test or correct a proposed optimal portfolio, and in this case imposing $\mathbf{J}^{\perp} = 0$ on the jumps is justified by mathematical elegance rather than economics.

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Region 1	USA	GE	JP	UK
R_i	0.1158	0.1155	0.1155	0.1191
J_i	-0.36			
Region 2	ARG	BR	CL	MX
R_i	0.1451	0.1452	0.1453	0.1454
J_i	-0.61			
Region 3	CH	HK	SNP	THA
R_i	0.1401	0.1402	0.1403	0.1404
J_i	-0.57			
	Variance Parameter			
v	0.15			
	Correlation Parameters			
ρ_0	0.15			
ρ	0.20			

Table I

Parameter Values for the Returns Processes.

This table reports the expected returns and jump size values used to calibrate the model for the three world regions and twelve countries.

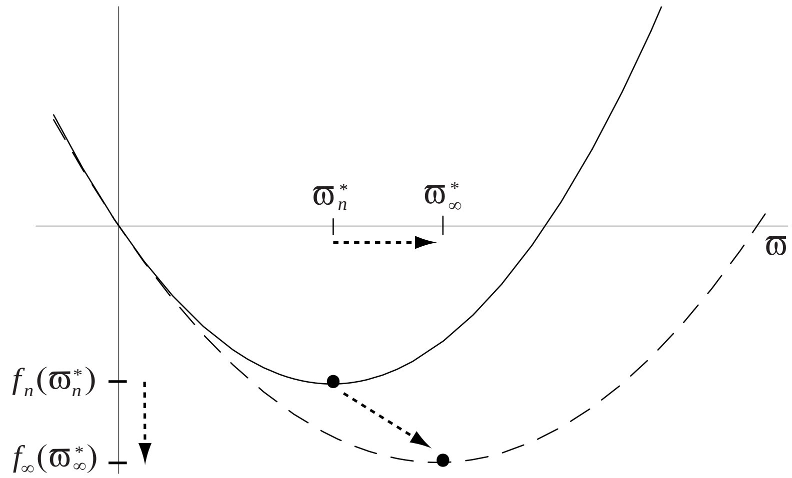

Figure 1
Scalar objective function used to determine the optimal portfolio weight ϖ_n^* and its large asset asymptotic limit, ϖ_∞^*.

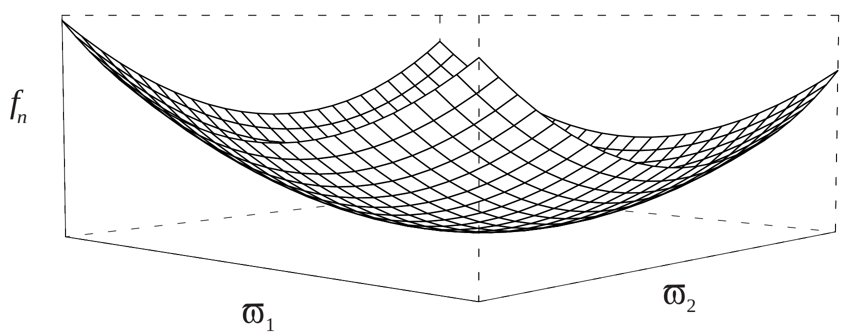

Figure 2
Bivariate objective function in a two-sector economy.

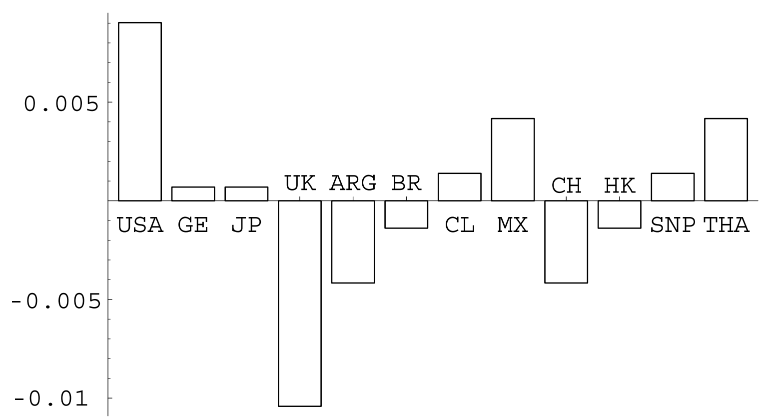

Figure 3

Optimal portfolio weights $\omega^{\perp *}$ for a 3-region world economy.

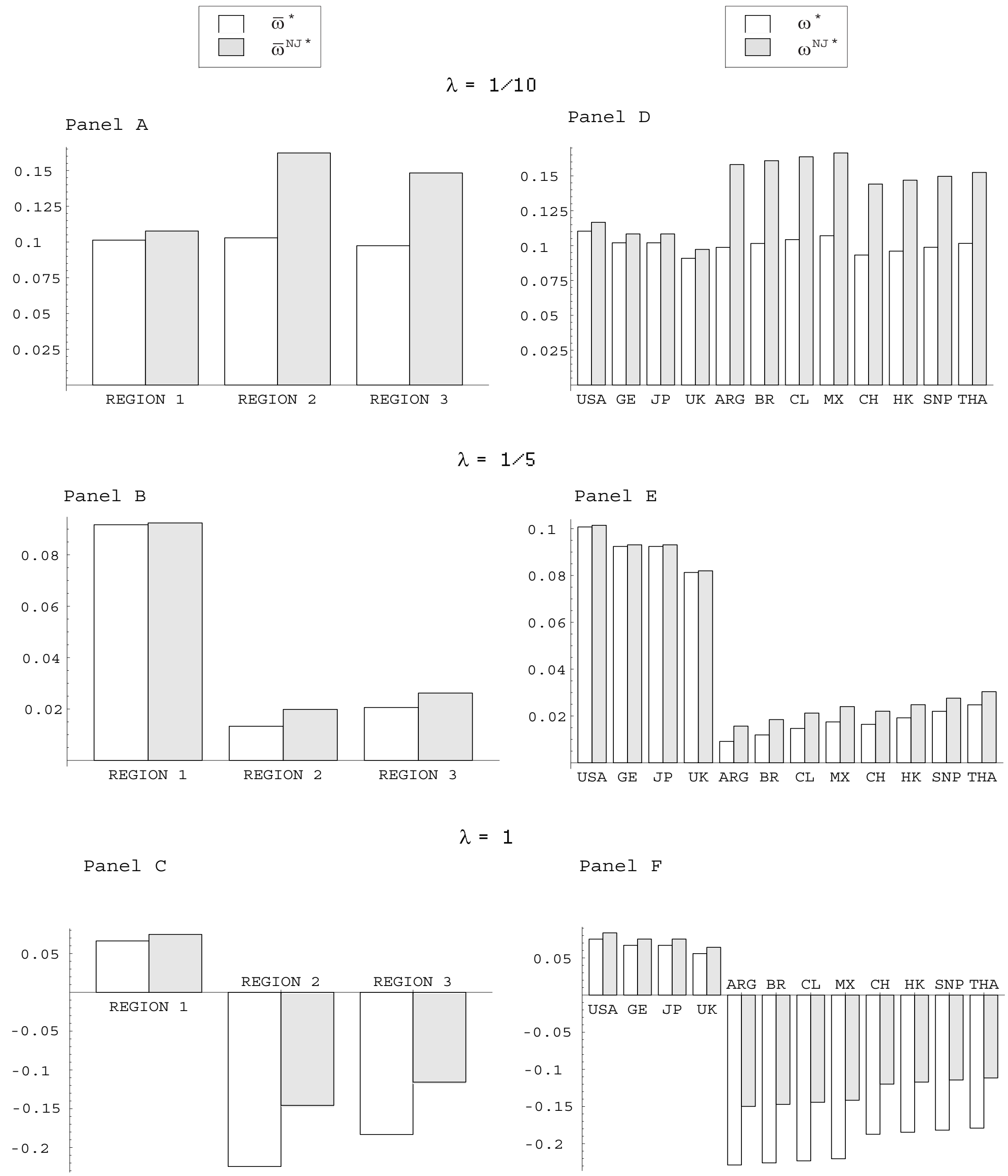

Figure 4

Optimal portfolio weights $\bar{\omega}^*$ and $\bar{\omega}^{NJ*}$ (Panels A, B, C) and ω^* and ω^{NJ*} (Panels D, E, F) for the 3-region world economy.