Robust Stochastic Discount Factors

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December 3, 2007

Overview

- Motivation and background
- Single period setting
- Numerical examples to give the idea
- Notions of Robustness
  - In expectation
  - In probability
- Continuous time model
- Example: stochastic volatility model

Motivation

- Pricing derivatives in an incomplete market
- The main challenges in asset pricing in an incomplete market
  - There are many stochastic discount factors (equivalent martingale measures, risk-neutral probability measures)
- Objective of this paper
  - Develop a simple theory for pricing derivatives in an incomplete market
  - The price of the derivative security is such that it is robust to potential model misspecification
  - It is also robust to misspecification of stochastic discount factor.

The Literature

- General Equilibrium approach:
- The representative agent approach: Lucas (1978), Cox, Ingersoll and Ross (1985), etc...
- Selection of a stochastic discount factor:
- model misspecification
- misspecification of the stochastic discount factors
- We use robustness as a criterion.

Road Map

- Notions of robustness in a single period setting.
- Two examples in a single period setting
- Notions of robustness in a continuous-time setting
- Main results
- Simple numerical illustration

Single Period Setting

Notions of Robustness in Single Period Setting

- There are four states at time one denoted by $(\omega_1, \omega_2, \omega_3, \omega_4)$. The probabilities of these states are $(1/3, 1/3, 1/6, 1/6)$.
- A single risky primitive asset and a riskfree bond.
- The payoff of the risky asset is described by a model $p = (8, 6, 3, 3)$.
- The payoffs of the risky and riskfree assets at the end of the period are as follows:

State	Payoff on risky asset	Payoff on riskfree asset
ω_1	8	1
ω_2	6	1
ω_3	3	1
ω_4	3	1

- The price of the risky asset is 4. The riskfree rate is zero.

Single Period Setting

- The market is incomplete. The stochastic discount factors are given by

$$m_\alpha :\equiv \left[3\left(\frac{3}{2}\alpha - 1\right), 3\left(2 - \frac{5}{2}\alpha\right), 3\alpha, 3\alpha\right],$$

where $\alpha \in (2/3, 4/5)$.

- The current price of the payoff p is given by

$$E_P[m_\alpha p] = E^\alpha[p]$$

- In this example

$$E_P[m_\alpha p] = 4 \; all \; m_\alpha$$

Robustness in Expectation

- Suppose there is call option with a strike price of $K = 7$ written on the risky asset.
- The no arbitrage price of the call, which is given by $\frac{3}{2}\alpha - 1$, falls in the set, $(0, 1/5)$.
- How should the price of this option be determined?
- Suppose the agents are not sure of the payoff vector.

Robustness in Expectation

- Consider this *special* perturbation of the risky asset's payoff.

$$p(\alpha,\epsilon) \equiv p + z_{\alpha,\epsilon} = \left(8 + \left(\frac{5}{2}\alpha - 2\right)\epsilon, 6 + \left(\frac{3}{2}\alpha - 1\right)\epsilon, 3, 3\right)$$

  for $\epsilon \in [0,1]$ with p as the benchmark payoff.

- $$\begin{aligned} E[m_\alpha p(\alpha,\epsilon)] &= E[m_\alpha p] + E[m_\alpha z_{\alpha,\epsilon}] \\ &= E[m_\alpha p] \\ &= E^\alpha[p] = 4 \end{aligned}$$

  So the market price of p does not change

Robustness in Expectation

Consider the band

$$\{|E^{\alpha}[\max\{p(\alpha,\epsilon_1)-7,0\}]-E^{\alpha}[\max\{p(\alpha,\epsilon_2)-7,0\}]|:\ \epsilon_1,\epsilon_2\in\mathcal{E}\}$$

The α that makes the above band the smallest is called the robust stochastic discount factor. However simpler to use a related metric.

Robustness in Expectation

- We pick α to minimize

$$\left\{ \left| E^{\alpha}[\max\{p(\alpha,\epsilon) - 7, 0\}] - \left(\frac{3}{2}\alpha - 1\right) \right| : \ \epsilon \in \mathcal{E} \right\}$$

- For any α, the pricing band of the call is given by

$$\left| E^{\alpha}[\max\{p(\alpha,\epsilon) - 7, 0\}] - \left(\frac{3}{2}\alpha - 1\right) \right| \in \left[0, \left(\frac{3}{2}\alpha - 1\right)\left(2 - \frac{5}{2}\alpha\right)\right]$$

- Robust call option pricing by choosing α.
- If we pick α just above $\frac{2}{3}$ or just below $\frac{4}{5}$ it will be robust to model misspecifaction

Robustness in Probability

- Next assume that the agents view the payoffs 8 and 6 in states ω_1 and ω_2, respectively, as being very close so that the agents may view the payoff vector $p = (8, 6, 3, 3)$ as the result of a small perturbation of the benchmark payoff vector whose first and second components are identical, i.e., a perturbation of a binomial tree.
- Consider the family of perturbations of the benchmark payoff given by

$$p(\alpha, \epsilon) = \left(8 + \left(\frac{5\alpha - 4}{1 - \alpha}\right)(1 - \epsilon), 6 + \left(\frac{3\alpha - 2}{1 - \alpha}\right)(1 - \epsilon), 3, 3\right)$$

  for $\epsilon \in [0, 1]$.
  - When $\epsilon = 1$, $p(\alpha, 1) = p$ which is the payoff of the risky asset.
  - When $\epsilon = 0$, $p(\alpha, 0) = \left(\frac{4-3\alpha}{1-\alpha}, \frac{4-3\alpha}{1-\alpha}, 3, 3\right)$, which has the same first and second components, required of the benchmark payoff vectors.
- Note that the benchmark in this case is a function of α.

Robustness in Probability

- Under the benchmark payoffs, the price of the option is

$$C(\alpha, 0) \equiv E^{\alpha}[\max\{p(\alpha, 0) - K, 0\}] = 4 - 3\alpha - K(1 - \alpha),$$

for $3 < K \leq (4 - 3\alpha)/(1 - \alpha)$.

- Let $\mathcal{F}$ denote the information set generated by the events, $\{\omega_1, \omega_3\}$ and $\{\omega_2, \omega_4\}$. Let $C(\alpha, \epsilon)$ denote the option price conditional on the events.

- Then,

$$C(\alpha, \epsilon) = E^{\alpha}\left[\max\{p(\alpha, \epsilon) - K, 0\} | \, \mathcal{F}\right],$$

and the option price at time 0 is given by $C_0(\alpha, \epsilon) = E^{\alpha}[C(\alpha, \epsilon)]$.

Robustness in Probability

- For the example with $K = 6$,

$$C(\alpha, 0) = 3\alpha - 2.$$

- The conditional option prices are given by
  - in the event $\{\omega_1, \omega_3\}$,

$$C(\alpha, \epsilon) = (3\alpha - 2) + \left[\left(1 + (1 - \epsilon)\frac{5\alpha - 4}{2(1 - \alpha)}\right)\frac{1}{2\alpha - 1} - 1\right](3\alpha - 2)$$

  - in the event $\{\omega_2, \omega_4\}$

$$C(\alpha, \epsilon) = (3\alpha - 2) + \left[(1 - \epsilon)\frac{4 - 5\alpha}{4(1 - \alpha)^2} - 1\right](3\alpha - 2),$$

Robustness in Probability

- Band of probabilities,

$$\{P^{\alpha}(\{|C(\alpha,\epsilon_1)-C(\alpha,\epsilon_2)|>\delta\}):\epsilon_1,\epsilon_2\in\mathcal{E}\}$$

  indexed by α.

- The α that gives the smallest probability band is called the robust stochastic discount factor (robust in probability).

- Band of probabilities,

$$\{P^{\alpha}(\{|C(\alpha,\epsilon)-C(\alpha,0)|>\delta\}):\epsilon\in\mathcal{E}\}$$

  indexed by α. Probability of large deviation.

- For the example, the probability band is the smallest when $\alpha\to 2/3$.

Notions of Robustness in Single Period Setting

- More formally, the payoff p may be misspecified and that the actual payoff is a payoff p_0 plus a perturbation/noise $z_{\alpha,\epsilon}$ in a family of possible perturbations indexed by $\alpha \in A$ and $\epsilon \in \mathcal{E}$.
- We assume that $\mathcal{E}$ is a segment in the real line including zero and that for all α, $\lim_{\epsilon \to 0} z_{\alpha,\epsilon} = 0$.
- We call p_0 the benchmark model or the benchmark payoff and
- $p_0 + z_{\alpha,\epsilon}$ the perturbed model or the perturbed payoff.
- We require that

$$p = p_0 + z_{\alpha,\epsilon}$$

for some $(\alpha, \epsilon) \in A \times \mathcal{E}$ and that for all $\alpha \in A$ and $\epsilon \in \mathcal{E}$,

$$E^{\alpha}[z_{\alpha,\epsilon}] = 0.$$

Notions of Robustness in Single Period Setting

- Allowing p_0 to depend on α,

$$p = p_0(\alpha) + z_{\alpha,\epsilon}$$

for some $(\alpha, \epsilon) \in A \times \mathcal{E}$ and that for all $\alpha \in A$ and $\epsilon \in \mathcal{E}$,

$$E^{\alpha}[p_0(\alpha) + z_{\alpha,\epsilon}] = E^{\alpha}[p_0(\alpha)].$$

- There are two approaches to formalizing that notion of robustness.
- One is in terms of prices. That is, we will find the stochastic discount factor m_{α^*} so that the pricing error band in the derivative security,

$$\{|E^{\alpha}[\phi(p_0(\alpha) + z_{\alpha,\epsilon_1})] - E^{\alpha}[\phi(p_0(\alpha) + z_{\alpha,\epsilon_2})]| : \epsilon_1, \epsilon_2 \in \mathcal{E}\}$$

is the smallest. The price of the derivative security $\phi(p)$ is then $E^{\alpha^*}[\phi(p)]$.

Notions of Robustness in Single Period Setting

- The expression

$$\{|E^{\alpha}[\phi(p_0(\alpha) + z_{\alpha,\epsilon_1})] - E^{\alpha}[\phi(p_0(\alpha) + z_{\alpha,\epsilon_2})]| : \epsilon_1, \epsilon_2 \in \mathcal{E}\} \quad (1)$$

  has clear intuitive appeal.

- However, it is often analytically more advantageous to evaluate the following band,

$$\{|E^{\alpha}[\phi(p_0(\alpha))] - E^{\alpha}[\phi(p_0(\alpha) + z_{\alpha,\epsilon})]| : \epsilon \in \mathcal{E}\} \quad (2)$$

- By a judicial choice of a benchmark model, the band (2) is not only easier to estimate than (1), but also a good approximation of the band (1).

Notions of Robustness in Single Period Setting

- The second approach is based on a band in probabilities.

$$\{P^{\alpha}(\{|E^{\alpha}[\phi(p_0(\alpha)+z_{\alpha,\epsilon_1})|\mathcal{F}]-E^{\alpha}[\phi(p_0(\alpha)+z_{\alpha,\epsilon_2})|\mathcal{F}]|>\delta\}):\ \epsilon_1,\epsilon_2\in\mathcal{E}\} \tag{3}$$

  for a given $\delta > 0$ and an information set $\mathcal{F}$.

- The objective is to find the stochastic discount factor such that the band is the smallest.

- By exactly the same reason as for the robustness in expectation approach, it is often analytically advantageous to examine the band

$$\{P^{\alpha}(\{|E^{\alpha}[\phi(p_0(\alpha)+z_{\alpha,\epsilon})|\mathcal{F}]-E^{\alpha}[\phi(p_0(\alpha))]|>\delta\}):\epsilon\in\mathcal{E}\} \tag{4}$$

  instead of (3).

Notions of Robustness in Single Period Setting

- Intuition — option pricing in a world with stochastic volatility.
  - Assume a setting as in Hull and White (1987) where the underlying stock price follows a geometric Brownian motion with a stochastic volatility.
  - Let V be the average variance, then $C(S_0, \sqrt{V})$.
  - In other words, $E^\alpha[\max\{S_T - K, 0\}|V] = C(S_0, \sqrt{V})$.
  - Moreover, given a market price of risk indexed by α, the price of the option under stochastic volatility is given by

$$E^\alpha[E^\alpha[\max\{S_T - K, 0\}|V]] = E^\alpha[C(S_0, \sqrt{V})].$$

  - Now suppose that the stochastic volatility process cannot be estimated precisely so that the price of the option under any particular potentially misspecified stochastic volatility model is $E^\alpha[C(S_0, \sqrt{V(\epsilon)})]$.

– In this context, if we take $\mathcal{F}$ as the information on the average variance, then $E^{\alpha}[\phi(p_0(\alpha) + z_{\alpha,\epsilon})|\mathcal{F}]$ in (3) is $C(S_0, \sqrt{V(\epsilon)})$ which is the price of the option under the stochastic discount factor indexed by α and conditional knowing the average variance that comes from a potentially misspecified stochastic volatility model indexed by ϵ.

Notions of Robustness in Single Period Setting

- Intuition Continued
  - Next consider the probability in (3), which in current context is,

$$P^{\alpha}\left(\left\{\left|C(S_0, \sqrt{V(\epsilon_1)}) - C(S_0, \sqrt{V(\epsilon_2)})\right| > \delta\right\}\right) \tag{5}$$

    where ϵ_1 and ϵ_2 are two arbitrary indices in $\mathcal{E}$.
  - Intuitively, if this probability is large, then the option prices under the two stochastic volatility models are very different.
  - Thus if the probability band in (3) is large, the option price is sensitive to potential model misspecification.
  - Put differently, if a stochastic discount factor can be chosen such that this probability band is the smallest, then under that stochastic discount factor, the option price is the least sensitive and hence robust to potential model misspecifications.

Continuous Time Setting

- We assume there is a single risky asset and a riskfree asset. The price dynamics of the risky asset follows the basic Black Scholes assumptions,

$$\frac{dS(t)}{S(t)} = \mu dt + \sqrt{v(t)}dW_1(t), \qquad S(0) = 1, \tag{6}$$

  where W_1 is a one-dimensional Brownian motion.

- We also assume that the riskfree rate is constant.
- The square of the volatility, $v(t)$ will be the main focus. Different specifications of it will lead to different models.
- It turns out that all the benchmark models and perturbed models differ only in the specification of the variance process.

Deterministic Volatility

- We assume the square of the volatility, $v(t)$, of the risky asset is a deterministic function of time and that it evolves towards some long run level,

$$dv(t) = (\kappa_1 - \kappa_2 v(t))dt, \quad v_0 = \sigma_0^2 \tag{7}$$

- We assume $\kappa_1 \geq 0$ and $\kappa_2 \geq 0$ where κ_2 is the speed of reversion and $\bar{v} = \kappa_1/\kappa_2$ is the long run target level.

Stochastic Volatility

- Stochastic volatility models.

$$dv(t) = (\kappa_1 - \kappa_2 v(t))dt + \sigma dW_2(t), \ v_0 = \sigma_0^2. \tag{8}$$

  where W_2 is a one-dimensional Brownian motion that is independent of W_1. Here κ_1, κ_2 and $\sigma > 0$ are model parameters. The instantaneous variance is $|v(t)|$ and the instantaneous volatility is $\sqrt{|v(t)|}$.

- The random noise $W_2(t)$ makes the market incomplete because the two sources of risk cannot be hedged with the risky asset and the riskfree asset.

Continuous Time Setting

- There exist another two dimensional Brownian motion

$$W^{(a,b)}(t) \equiv (W_1^{(a,b)}(t), W_2^{(a,b)}(t))$$

  such that the risky asset's dynamics under this measure satisfies the following stochastic differential equations:

$$\begin{aligned} \frac{dS(t)}{S(t)} &= r dt + \sqrt{|v(t)|} dW_1^{(a,b)}(t), \qquad S(0) = 1, \\ dv(t) &= [\kappa_1 - \kappa_2 v(t) - \sigma(a + bv(t))]dt + \sigma dW_2^{(a,b)}(t). \end{aligned}$$

- The relationship between the two Brownian motions is given by the Girsanov transformation

$$W_1^{(a,b)} = W_1(t) + \int_0^t \frac{\mu - r}{\sqrt{v(s)}} ds; \quad W_2^{(a,b)}(t) = W_2(t) + \int_0^t (a + bv(s)) ds$$

Perturbed Models

- We now describe the perturbed models. Fix a market price of risk. Consider the following family of stochastic volatility model indexed by (a, b) and ϵ, under the probability measure $P^{(a,b)}$,

$$\begin{aligned} \frac{dS(t)}{S(t)} &= rdt + \sqrt{|v(t)|}\, dW_1^{(a,b)}(t), \qquad S(0) = 1, \\ dv(t) &= [\kappa_1 - \kappa_2 v(t) - \sigma(a + bv(t))]\, dt + \epsilon\sigma dW_2^{(a,b)}(t). \end{aligned}$$

- It is useful to introduce a change of notation.

$$\begin{aligned} \frac{dS(t)}{S(t)} &= rdt + \sqrt{|v(t;\kappa_1,\kappa_2,\epsilon)|}\, dW_1^{(\kappa_1,\kappa_2)}(t), \qquad S(0) = 1, \\ dv(t;\kappa_1,\kappa_2,\epsilon) &= [\kappa_1 - \kappa_2 v(t;\kappa_1,\kappa_2,\epsilon)]\, dt + \epsilon\sigma dW_2^{(\kappa_1,\kappa_2)}(t). \end{aligned}$$

Continuous Time Setting

- We will focus on the deterministic volatility benchmark. Under the benchmark model, the stock price is given by

$$\begin{aligned}\frac{dS(t)}{S(t)} &= r dt + \sqrt{v(t;\kappa_1,\kappa_2)}\, dW_1^{(\kappa_1,\kappa_2)}(t), \qquad S(0)=1,\\ dv(t;\kappa_1,\kappa_2) &= [\kappa_1 - \kappa_2 v(t;\kappa_1,\kappa_2)]dt, \quad v(0;\kappa_1,\kappa_2)=\sigma_0^2,\end{aligned}$$

  where $\kappa_1 > 0$ and $\kappa_2 > 0$.

- Since $v(t;\kappa_1,\kappa_2)$ is deterministic, we can use no arbitrage arguments to obtain a *unique* price for any derivative written on the underlying asset.

Continuous Time Setting

- Let
$$V(\kappa_1, \kappa_2, 0) = \int_0^T v(t; \kappa_1, \kappa_2) dt$$

- The price of a call option with strike price K and time to maturity T on the risky asset is
$$\Phi(d_2)$$

$$d_1 = \frac{\ln(S(0)/Ke^{-rT}) + \frac{1}{2}V(\kappa_1, \kappa_2, 0)}{\sqrt{V(\kappa_1, \kappa_2, 0)}}, \qquad d_2 \equiv d_1 - \sqrt{V(\kappa_1, \kappa_2, 0)}$$

and $\Phi(\cdot)$ is the standard normal cumulative distribution function.

Continuous Time Setting

- Under the perturbed model, let

$$V(\kappa_1, \kappa_2, \epsilon) = \int_0^T |v(t; \kappa_1, \kappa_2, \epsilon)| dt.$$

The price of the call option is equal to

$$E^{(\kappa_1,\kappa_2)} \left[C_{BS} \left(\sqrt{V(\kappa_1, \kappa_2, \epsilon)} \right) \right],$$

Continuous Time Setting — Robustness in Expectation

- We approximate the difference

$$E^{(\kappa_1,\kappa_2)}\left[C_{BS}\left(\sqrt{V(\kappa_1,\kappa_2,\epsilon)}\right)\right] - C_{BS}\left(\sqrt{V(\kappa_1,\kappa_2,0)}\right),$$

  by a second order Taylor expansion

- Define

$$F(\kappa_1,\kappa_2) = \lim_{\epsilon\sigma\downarrow 0} \frac{E^{(\kappa_1,\kappa_2)}\left[C_{BS}\left(\sqrt{V(\kappa_1,\kappa_2,\epsilon)}\right)\right] - C_{BS}\left(\sqrt{V(\kappa_1,\kappa_2,0)}\right)}{\epsilon^2\sigma^2}.$$

Continuous Time Setting — Robustness in Expectation

Proposition One

$$F(\kappa_1,\kappa_2) = \left| \frac{C''_{BS}\left(\sqrt{V(\kappa_1,\kappa_2,0)}\right)}{8V(\kappa_1,\kappa_2,0)} - \frac{C'_{BS}\left(\sqrt{V(\kappa_1,\kappa_2,0)}\right)}{8(V(\kappa_1,\kappa_2,0))^{3/2}} \right| \times \int_0^T \left[\frac{1-e^{-\kappa_2(T-u)}}{\kappa_2}\right]^2 du.$$

The second order approximation of the band in

$$E^{(\kappa_1,\kappa_2)}\left[C_{BS}\left(\sqrt{V(\kappa_1,\kappa_2,\epsilon)}\right)\right] - C_{BS}\left(\sqrt{V(\kappa_1,\kappa_2,0)}\right),$$

is then $F(\kappa_1,\kappa_2)$ multiplied by the square of the length of $\sigma\mathcal{E}$.

Continuous Time Setting — Robustness in probability

Proposition Two

For all $\delta \in (0, \delta_0(\kappa_1, \kappa_2)]$ where $\delta_0(\kappa_1, \kappa_2) = \sqrt{V(\kappa_1, \kappa_2, 0)}$, and for all $0 < c < 1$, we have

$$\begin{aligned} &\ln P^{(\kappa_1,\kappa_2)} \left[\left| C_{BS}\left(\sqrt{V(\kappa_1, \kappa_2, \epsilon)}\right) - C_{BS}\left(\sqrt{V(\kappa_1, \kappa_2, 0)}\right) \right| \geq \delta \right] \\ \leq\ & -\frac{c}{\epsilon^2 \sigma^2} J(\kappa_1, \kappa_2, \delta), \end{aligned}$$

as $\epsilon\sigma \downarrow 0$, where

$$\begin{aligned} J(\kappa_1, \kappa_2, \delta) &= J_1(\kappa_2)\left(2\sqrt{V(\kappa_1, \kappa_2, 0)}\delta - \delta^2\right)^2, \quad \delta \leq \delta_0(\kappa_1, \kappa_2), \\ J_1(x) &= \frac{x^3(e^{2xT} - 1)}{(e^{\frac{x}{2}T} - e^{-\frac{x}{2}T})^4}. \end{aligned}$$

Continuous Time Setting — Robustness in probability

- An immediate implication of Proposition is that the probability band

$$\left\{ P^{(\kappa_1,\kappa_2)} \left[\left| C_{BS}\left(\sqrt{V(\kappa_1,\kappa_2,\epsilon)}\right) - C_{BS}\left(\sqrt{V(\kappa_1,\kappa_2,0)}\right) \right| \geq \delta \right] : \ \epsilon \in \mathcal{E} \right\}$$

is bounded by exp $\left(-\frac{c}{\epsilon^2\sigma^2} J(\kappa_1,\kappa_2,\delta)\right)$ at least for $\mathcal{E}$ that is small.

- Proposition is derived using the tools of large deviation theory.

Numerical Illustration — Robustness in probability

- Calibration of Parameters
- Recall that $J(\kappa_1, \kappa_2, \delta)$ measures the rate of convergence in probability (Proposition) given three parameters κ_1, κ_2 and δ. Thus our numerical study will focus mostly on the function $J(\kappa_1, \kappa_2, \delta)$.
- It will be useful to introduce a parameter η and a function H,

$$\eta = \frac{\delta}{H(\kappa_1, \kappa_2)}, \quad H(\kappa_1, \kappa_2) = C_{BS}\left(\sqrt{V(\kappa_1, \kappa_2, 0)}\right).$$

- The parameter is used to express the percentage price difference.

Numerical Illustration — Robustness in probability

- Formula

$$\begin{aligned}&\ln P^{(\kappa_1,\kappa_2)}\left[\left|C_{BS}\left(\sqrt{V(\kappa_1,\kappa_2,\epsilon)}\right)-C_{BS}\left(\sqrt{V(\kappa_1,\kappa_2,0)}\right)\right|\geq\delta\right]\\ \leq\ & -\frac{c}{\epsilon^2\sigma^2}J(\kappa_1,\kappa_2,\delta),\end{aligned}$$

can then be rewritten as

$$\begin{aligned}&P^{(\kappa_1,\kappa_2)}\left(\left|\frac{C_{BS}\left(\sqrt{V(\kappa_1,\kappa_2,\epsilon)}\right)-H(\kappa_1,\kappa_2)}{H(\kappa_1,\kappa_2)}\right|\geq\eta\right)\\ \leq\ & \exp\left\{-\frac{cJ\left(\kappa_1,\kappa_2,\eta H(\kappa_1,\kappa_2)\right)}{\sigma^2\epsilon^2}\right\},\end{aligned} \tag{9}$$

for all $0 < c < 1$ and all $\eta \leq \eta_0$ where $\eta_0 = \frac{\delta_0}{H(\kappa_1,\kappa_2)}$ and δ_0 is defined in Proposition .

- In the large deviation literature $\frac{1}{\sigma^2\epsilon^2} J\left(\kappa_1, \kappa_2, \eta H(\kappa_1, \kappa_2)\right)$ is called the rate function because it characterizes the rate at which the probability on the left hand side of (9) converges to zero.

Numerical Illustration — Robustness in probability

The following proposition records some properties of the rate function J which will be useful for our numerical analysis and for potential applications.

Proposition Three

(1) $J(\kappa_1, \kappa_2, \eta H(\kappa_1, \kappa_2))$ is increasing in η over the range $(0, \eta_0(\kappa_1, \kappa_2))$.

(2) $J(\kappa_1, \kappa_2, \eta H(\kappa_1, \kappa_2))$ is a decreasing function of the strike price K.

(3) For $\eta > 0$ and $\kappa_2 > 0$, $J(\bar{v}\kappa_2, \kappa_2, \eta H(\bar{v}\kappa_2, \kappa_2))$ is increasing in $\bar{v}$.

(4) For $\eta > 0$ and $\bar{v} > \sigma_0^2$, $J(\bar{v}\kappa_2, \kappa_2, \eta H(\bar{v}\kappa_2, \kappa_2))$ is increasing in κ_2.

Numerical Illustration — Robustness in Expectation

- the parameters are the same as earlier
- The following table reports the percentage pricing error bound

$$\frac{\sigma^2 F(\kappa_1,\kappa_2)}{C_{BS}\left(\sqrt{V(\kappa_1,\kappa_2,0)}\right)}$$

$$\approx \frac{E^{(\kappa_1,\kappa_2)}\left[C_{BS}\left(\sqrt{V(\kappa_1,\kappa_2,\epsilon)}\right)\right] - C_{BS}\left(\sqrt{V(\kappa_1,\kappa_2,0)}\right)}{C_{BS}\left(\sqrt{V(\kappa_1,\kappa_2,0)}\right)}.$$

- multiplied by 1000.

Pricing Error Bound

	$S(0)/K$							
	0.90	0.925	0.95	0.975	1.00	1.025	1.05	1.10
$(\bar{v}, \kappa_2) =$								
$(0.05, 1.0)$	10.646	5.826	3.624	2.543	1.863	1.295	0.774	0.040
$(0.05, 1.5)$	9.374	5.158	3.223	2.268	1.664	1.161	0.698	0.029
$(0.05, 2.0)$	8.299	4.589	2.879	2.032	1.493	1.044	0.631	0.020
$(0.05, 2.5)$	7.386	4.102	2.584	1.827	1.345	0.943	0.572	0.013
$(0.05, 3.0)$	6.604	3.683	2.327	1.650	1.216	0.854	0.521	0.007
$(0.10, 1.0)$	7.083	4.098	2.672	1.930	1.442	1.032	0.653	0.038
$(0.10, 1.5)$	5.351	3.177	2.114	1.547	1.167	0.845	0.548	0.058
$(0.10, 2.0)$	4.163	2.526	1.710	1.264	0.961	0.704	0.465	0.067
$(0.10, 2.5)$	3.318	2.050	1.408	1.050	0.804	0.594	0.399	0.071
$(0.10, 3.0)$	2.697	1.693	1.177	0.885	0.681	0.507	0.345	0.070

Conclusion

- Developed an approach to price a derivative security
- The price of the derivative security is such that it is robust to potential model misspecification
- It is also robust to misspecification of stochastic discount factor.
- Future work
  - develop a general equilibrium model
  - robust hedging