Deriving the implied term structure of default probabilities and recovery rates for each pair of industry and rating category via corporate bond data

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Basic Structure of Defaultable Bond Pricing

Three Fundamental Elements

- Process for Term Structure of Interest Rates :
- Default-Event Generation Process
- Recovery Rate Process

The Information Sources for Defaults and Credits :

・Backward-Looking：Statistical data on defaults, Markov transition model using past data

・Forward-Looking :CBs, DSs（default swap）, Stocks

Concept of Default and a priori model

- Consistency in the definition of default and modeling
- The CF structure of an enterprise depends on the portfolio of business lines associated with industry factors
- The concept of industry is relative.
- Business cycles, industry cycles and economic structure

The market is essentially incomplete.

- A default process in practice is in general non-Markovian.
- The recovery rate is determined after a long evaluation and negotiation process among those of interests, which is costly.
- Business cycles in each industry are often different, differently affecting each firms that have a various portfolio of business lines.

Price data set at *n* of corporate bonds delivers the information:

- Investor's evaluation on the term structure of credit risk in the cbs issued by firms that have different industry factors
- The evaluation includes their considerations on the industry portfolio structure of each firm.
- Hence prices at n of many cbs implicitly carry the investor's view on the **TSDP** (term structure of default probabilities) for each industry, provided the cb market is efficient.
- Here the industry concept is something common to investors in evaluation.
- Also, the prices often reflect the rating categories.

- In short, the information is investors' forward-looking evaluation on default structure of each pair of industry category and rating category for existing firms that have different portfolio of business lines.

In our modeling

- The CF structure of an enterprise depends on the portfolio of business lines associated with industry factors, where industry category is given in advance.
- We take into account the **business (industry) portfolio structure of each firm** and use the sales proportions of each industry business lines as a description of the portfolio weights.
- **Discount factors** for valuing the defaultable cash flows of cbs are derived by modeling gbs (government bonds).

Interest rate model:new developments

Our DF is stochastic through a relation with forward interest rate, which is **attribute-dependent** on coupon and maturity of bond (Convenience).

- Collin-Dufresne,P. &Solnik, B.(2001).On the term structure of default premia in the swap and LIBOR markets,JF
- Duffie,D. & Kan,R.(1996). A yield factor model of interest rate. Math F State Space model
- Feldhutter,P. &Lando,D.(2007). Decomposing swap spreads. **Convenience yields**

Pricing G-bonds (fixed coupon)

Let t be the present time and G *the* # of G-bonds. Let

$$t + s_{am} \qquad (m = 1, \cdots, Ma)$$

be all the combined CF time points at which some G bonds generate CF. Let $C_{gt}(s)$ be the CF function of the gth bond,

which is zero unless $s = s_{aj}$ and s belongs to the set of theCF points of the gth bond. Then usually

$$P_{gt}(1) = \sum_{j=1}^{Ma} C_{gt}(s_{gj}) \overline{\overline{D_t}}(s_{gj}) \qquad \overline{\overline{D_t}}(s_{gj}) = E_t[\exp(-\int_0^{s_{gj}} r_{t+s} ds)]$$

Where $\{r_s\}$ is a spot rate process.

$$\overline{\overline{D_t}}(s_{gj}) = H(r_t, s_{gj}, \theta) \qquad H(r_t, s, \theta) = \exp(-Rs)$$

Attribute-dependent formulation for interest rates

$$\overline{\overline{D}}_{gt}(s_{gj}) = E_t[\exp(-\int_0^{s_{gj}} r_{gt+s} ds)]$$

Attribute-dependent DF and forward rates

$$P_{gt}(2) = \sum_{j=1}^{Ma} C_{gt}(s_{gj}) D_t^*(s_{gj})$$

Here the DF is stochastically realized with a realization of the whole forward rate term structure $\{f_{ts}: s>0\}$

$$D_t^*(s_{aj}) = \exp(-\int_0^{s_{aj}} f_{ts} ds)$$

A realization of the term structure $\{D_t(s): s>0\}$ corresponds to a realization of term structure $\{f_{ts}: s>0\}$ in one-one manner.

$$D_{gt}^*(s_{gj}) = \exp(-\int_0^{s_{gj}} f_{gts} ds)$$

The term structure of forward rates $\{f_{gts} : \ s > 0\}$ is assumed to be dependent on the attributes of income (CF) structure $\{C_{gt}(s_{am})\}$:coupon and maturity.

Ie, investors see a convenience value in the CF structure and discount the CFs along the value. One may specify the forward rates, e.g.,

$$f_{gts} = f_{(1)ts} + f_{(2)gts},$$

for each CFs where the second term corresponds to a convenience yield. (Feldhutter,P. &Lando,D.(2007))

In Kariya and Tsuda (1994)(1995) the attribute-dependent DF is used in modeling government bonds and the empirical validity is shown by many examples. The effectiveness of the model is demonstrated for USGB (1997).

Using the attribute-dependent forward term structure,

$$P_{gt} = \sum_{m=1}^{Ma} C_{gt}(s_{am}) D_{gt}(s_{am})$$

$$D_{gt}(s) = \bar{D}_{gt}(s) + \Delta_{gt}(s)$$ Mean DF + Random DF

It is noted that this modeling is unconditional, while the modeling using spot rates in no-arbitrage argument is conditional.

$$P_{gt} = \sum_{m=1}^{M_a} C_{gt}(s_{am}) \bar{D}_{gt}(s_{am}) + \eta_{gt}$$

$$\eta_{gt} = \vec{C}'_{gt} \vec{\Delta}_{gt}$$

$$\vec{C}_{gt} = (C_{gt}(s_{a1}), \cdots, C_{gt}(s_{aM_a}))' : Ma \times 1$$

$$\vec{\Delta}_{gt} = (\Delta_{gt}(s_{a1}), \cdots, \Delta_{gt}(s_{aM_a}))' : Ma \times 1$$

The attribute-dependent mean DF is approximated by polynomial

$$\bar{D}_{gt}(s) = 1 + (\delta_{11t} z_{g1t} + \delta_{12t} z_{g2t})s + \cdots + (\delta_{pt} z_{gp1t} + \delta_{p2t} z_{g2t})s^{p}$$

z_{g1t} : coupon rate z_{g2t} : maturity

The specification of the covariance structure of the stochastic parts is important in take into account the correlation structure.

$$Cov(\vec{\Delta}_{gt}, \vec{\Delta}_{ht}) = \lambda_{ght} \Phi_{ght}$$

$$\Phi_{ght} = (\phi_{ght \cdot jr})$$

$$\phi_{ght \cdot jr} = \exp(-|s_{aj} - s_{ar}|)$$

$$\lambda_{ght} = \begin{cases} \sigma^{2} & (g = h) \\ \sigma^{2} \rho b_{ght} & (g \neq h) \end{cases}$$

This specification implies

$$Cov(D_{gt}(s_{aj}), D_{ht}(s_{ar})) = \lambda_{ght}\phi_{ght\cdot jr}$$

$$Cov(P_{gt}, P_{ht}) = Cov(\eta_{gt}, \eta_{ht}) = \lambda_{ght}\vec{C}'_{gt}\Phi_{ght}\vec{C}_{ht} \equiv f_{ght}$$

1) The closer the two time points s_{aj}, s_{ar} of income cash flows, the greater the correlation is.
2)The greater the difference of the maturity periods of two bonds, the less the correlation of the two prices is.

These two effects are combined and the unknown parameters are estimated cross-sectionally and simultaneously together with the parameters of the mean DF.

As in Kariya and Tsuda(1994), this specification of the covariance structure empirically models government bond prices very well not only for JGB but for USGB

CB Pricing Model

$$V_{kt} = \sum_{j=1}^{M(k)} \tilde{C}_{kt}(s_{kj}) D_{kt}(s_{kj})$$

$$D_{kt}(s) = \overline{D}_{kt}(s) + \Delta_{kt}(s)$$

The mean $\overline{D}_{kt}(s)$ is the same as that of gb and attribute-dependent , but $\Delta_{kt}(s)$ depends on the variational structure of cbs.

The defaultable income cash flows are stochastically expressed:

$$\tilde{C}_{kt}(s_{kj}) = C_{kt}(s_{kj})(1 - L_{kt+s_{kj}}) + 100\gamma(i(k))L_{kt+s\,kj}(1 - L_{kt+s_{kj}-1})$$

where the default event process is expressed by default time;

$$L_{kt+s} = \begin{cases} 0 & if \quad J_k > t+s \\ 1 & if \quad J_k \le t+s \end{cases}$$

Mean CFs

$$\overline{C}_{kt}(s_{kij}) = C_{kt}(s_{kj})[1 - p_{kt}(s_{kj} : i(k))]$$
$$+100\gamma(i(k))[p_{kt}(s_{kj} : i(k)) - p_{kt}(s_{kj-1} : i(k))]\chi_{kt}(s_{kj})$$

Coupon ∗ [prob of no-default till s_{kj}] +
(face value) ∗ (recovery rate) ∗ [default prob in (s_{kj-1}, s_{kj}]]

Decomposition of default probability into $p_t(s : i, j)$
(rating, industry)
and industry weights $w_k(j) \geq 0, \quad \sum_{j=1}^{J} w_k(j) = 1$

$$p_{kt}(s : i(k)) \equiv \sum_{j=1}^{J} w_k(j)\, p_t(s : i(k), j)$$

TSDP

$$p_t(s : i, j) = \alpha_{1t}^{ij} s + \alpha_{2t}^{ij} s^2 + \cdots + \alpha_{qt}^{ij} s^q$$

Then the model is expressed as a regression model with error terms being heavily dependent

$$V_{kt} = \sum_{m=1}^{Ma} \bar{C}_{kt}(s_{am})\overline{D_t}(s_{am}) + \varepsilon_{kt}$$

$$\varepsilon_{kt} = \sum_{j=1}^{Ma} \bar{C}_{kt}(s_{aj})\Delta_{kt}(s_{aj})$$

We assume the same covariance structure for Δ as before. This error term depends on the parameters α's of default prob fn and may be approximated as

$$\varepsilon_{kt}' = \sum_{m=1}^{M} C_{kt}(s_{am})\Delta_{kt}(s_{am})$$

,which we assume. The mean (theoretical) value of a CB is

$$\overline{V_{kt}} = \sum_{m=1}^{Ma} \bar{C}_{kt}(s_{am})\overline{D_t}(s_{am})$$

where $\overline{D}_t(s)$ is the attribute-dependent DF derived from gbs.

Then the covariance of the *k*-th and *l*-th cb prices is

$$Cov(\varepsilon_{kt}, \varepsilon_{lt}) = \lambda_{i(k)i(l)t}\phi_{klt}$$

$$\phi_{klt} = \vec{C}'_{kt}\Phi_{klt}\vec{C}_{lt}$$

$$\lambda_{i(k)i(l)t} = \begin{cases} \sigma^2 & (k = l) \\ \sigma^2\rho_{i(k)i(l)}b_{klt} & (k \neq l) \end{cases}$$

Here $\rho_{i(k)i(l)}$ denotes the correlation between the cb prices of the ratings *i*(*k*) and *i*(*l*), which is assumed to be constant for each pair of ratings: if the two cbs are of the same rating, it is $\rho^{(i)}$

$$\rho_{i(k)i(l)} = \begin{cases} \rho^{(i)} & (i(k) = i(l) = i) \\ \xi \cdot \exp(-\theta \,|\, i(k) - i(l) \,|) & (i(k) \neq i(l)) \end{cases}$$

The more distant the ratings are, the less the correlation is.

One may assume

$$\rho_{i(k)i(l)} = \begin{cases} \rho^i & (i(k) = i(l) = i) \\ 0 & (i(k) \neq i(l)) \end{cases}$$

Estimation procedure

1)For each rating category i, let $k_1, \cdots, k_n$ be the cb#s of rating i, and for each pair (ρ^i, γ^i) of the correlation and recovery rate, we estimate $\{\alpha_l^{ij}\}$ by the GLS, where $\rho^i = \rho(i)$ and $\gamma^i = \gamma(i)$ move over {h/100:h=0,1,…,99} and (ρ^{i*}, γ^{i*}) minimizing the generalized squares is selected this optimizer is fixed and used in the next step. But the GLSE for $\{\alpha_l^{ij}\}$ obtained here is not used.

2) Repeat this for *i*=1,…*I* to get all the (ρ^{i*}, γ^{i*}).

3)To estimate $\{\alpha_l^{ij}\}$ simultaneously by using all the cb prices, we use the GLS for each given pair of (ξ, θ) moving over {h/100:h=0,1,…,99} to get optimum values $\{\alpha_l^{ij*}\}$ $(\xi*, \theta*)$

4) The above procedure is repeated for some orders of the polynomial of default probability $p_t(s : i, j)$.

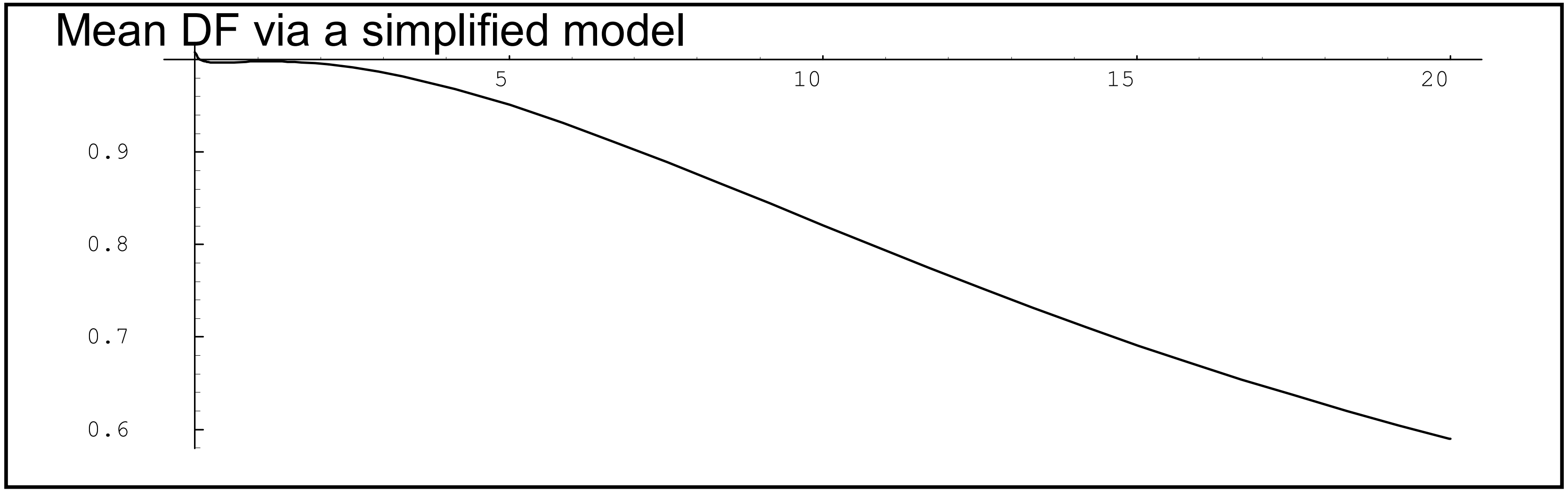
$$p_t(s : i, j) = \alpha_{1t}^{ij} s + \alpha_{2t}^{ij} s^2 + \cdots + \alpha_{qt}^{ij} s^q$$

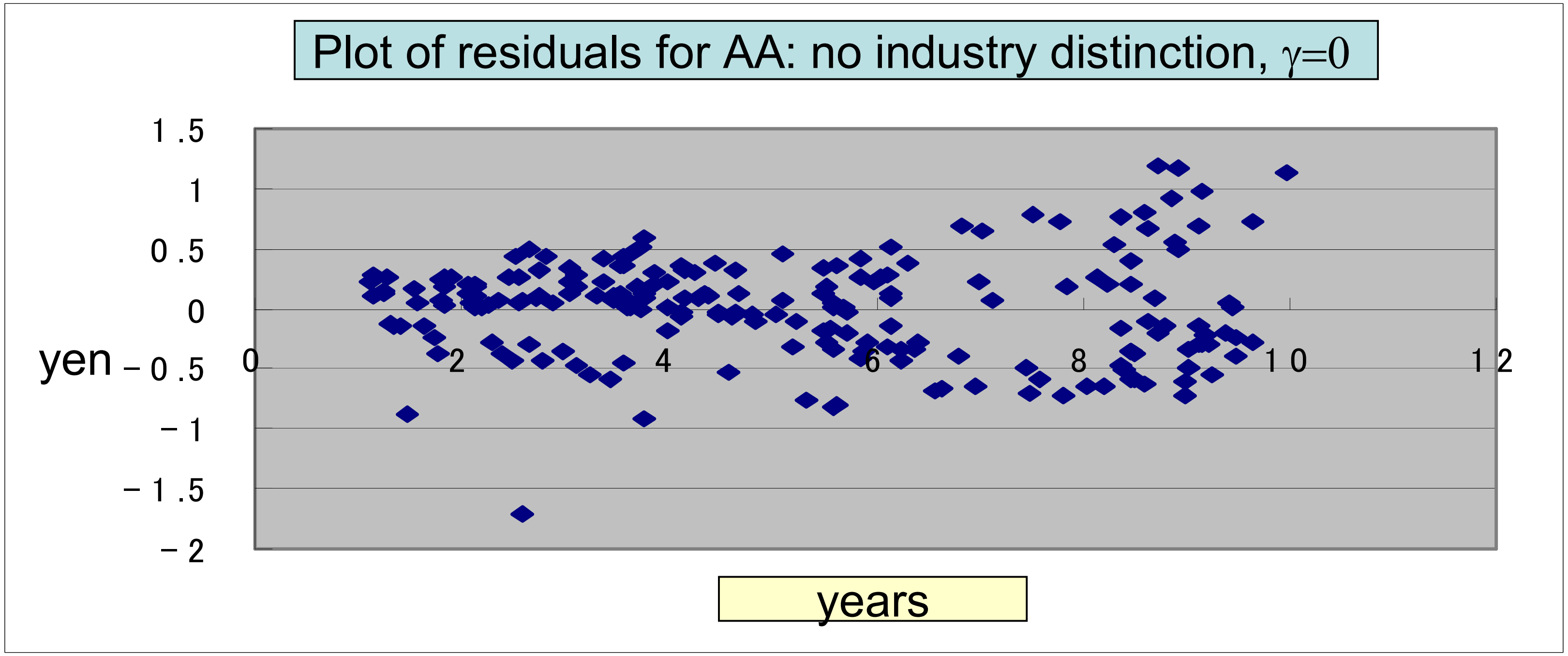
Thus the TSDP is obtained for each pair of rating category and industry category.

The validity of this modeling and estimation is checked by comparing the observed and estimated values of cb prices:

$$\sqrt{\frac{1}{N} \sum_{n=1}^{N} (V_{nt} - \widehat{V}_{nt})^2}$$

We need a thorough empirical work; but so far not yet.


Mean DF via a simplified model
5
10
15
20
0.9
0.8
0.7
0.6


Plot of residuals for AA: no industry distinction, γ=0
1.5
1
0.5
0
yen
-0.5
-1
-1.5
-2
0
2
4
6
8
10
12
years

Summary

- We propose a pricing model for cb that derives the implied TSDP for each pair of elements in rating and industry categories.
- In decomposing the stochastic DF for cash flows, the mean part is the same as the attribute-dependent mean DF derived by gb prices, while the stochastic part is of the variational structure associated with that of cb prices.
- Estimation procedure is based on the 3 stage GLS;
- 1)GLS with gb prices that determines the mean DF,
- 2)GLS with cb prices in each rating category that determines recovery rate and correlation in each rating category
- 3)GLS with all the cb prices that determines the TSDP for each pair of rating and industry categories.
- A partial empirical result and Kariya and Tsuda' result will support the effectiveness model.

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