Prepayment and Mortgage Rate Modeling

Recent Advances in Mathematical Finance

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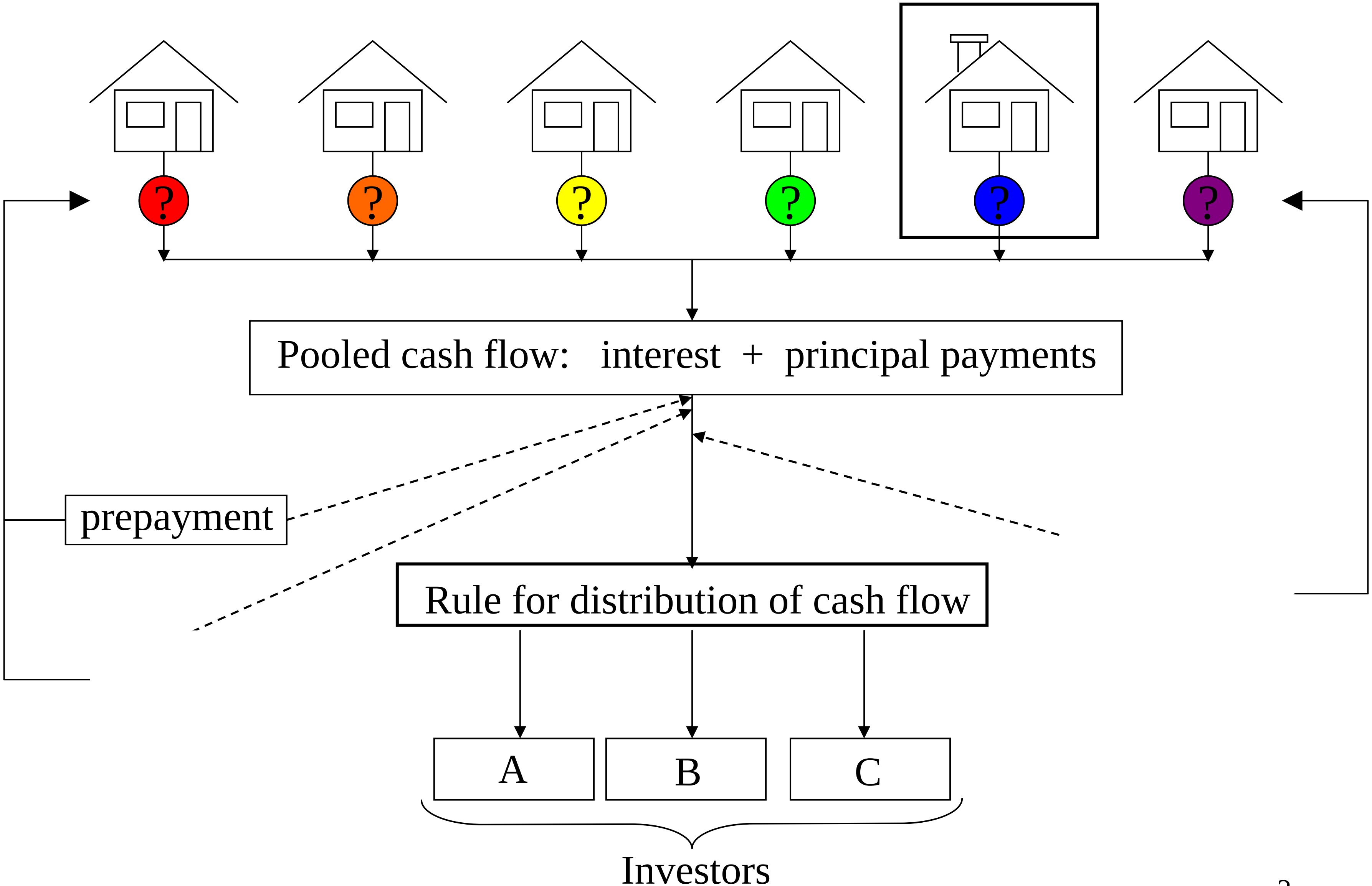
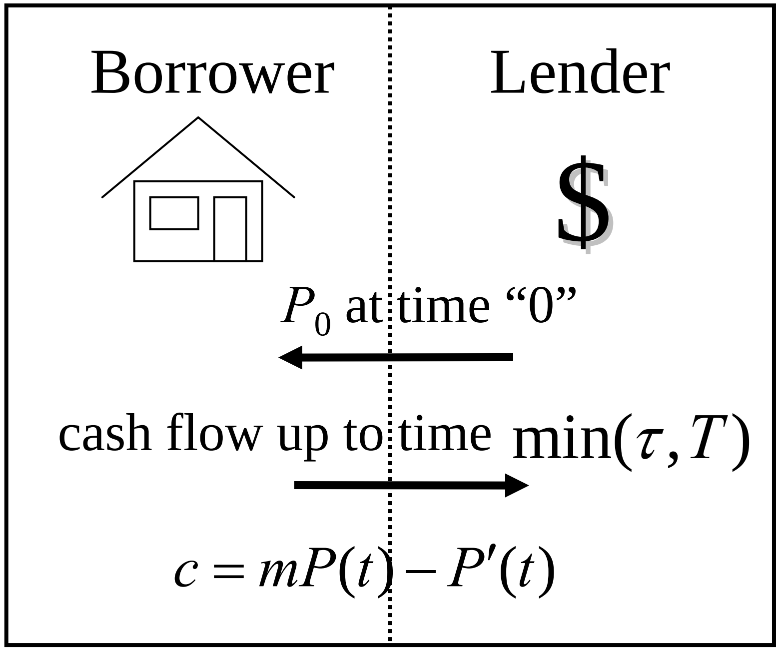
Mortgage Securities



A Mortgage



- m – mortgage rate
- $P(t)$ – outstanding principal
- γ_t – Φ_t-intensity of prep. ("prepayment rate")
- c – payment rate (**$**/time)
- τ – prepayment time
- Φ_t – "relavant" information

$$M_t = P(t) + \mathbf{E}\left[\int_t^T P(s)\left(m - r_s\right) e^{-\int_t^s r_\theta + \gamma_\theta d\theta} ds \,\middle|\, \mathrm{F}_t \right]$$

Prepayment (Intensity) Specification

Prepayment: Empirical or Model-based

$$\gamma_t := \quad \gamma(X_t) \quad \text{or} \quad \gamma(\Pi_t)$$

market factors

refinancing incentive

$$\Pi(X_t)$$

measure of *borrower*'s reaction

$$\Pi_t := \begin{cases} m^0 - m^t, & \text{comparison of mortgage rates} \\ L_t - P(t), & \text{"market price" of profitability} \end{cases}$$

Note: the incentive translates "*market*" to "*resident*" money-language

Prepayment

or

Intensity $\gamma_t := \gamma(\Pi_t)$ Modeling

Intensity function $\gamma(\cdot)$

(“*Low frequency”*)

Estimates the borrower’s “response” (in probabilistic terms) to certain market situations.

Refinancing incentive Π_t

(“*High frequency”*)

Estimates “usefulness” of the refinancing from the borrower’s point of view.
Based on the “*market*” information (interest, unemployment rates, home prices).

Why to *Model*?

sensitive to time

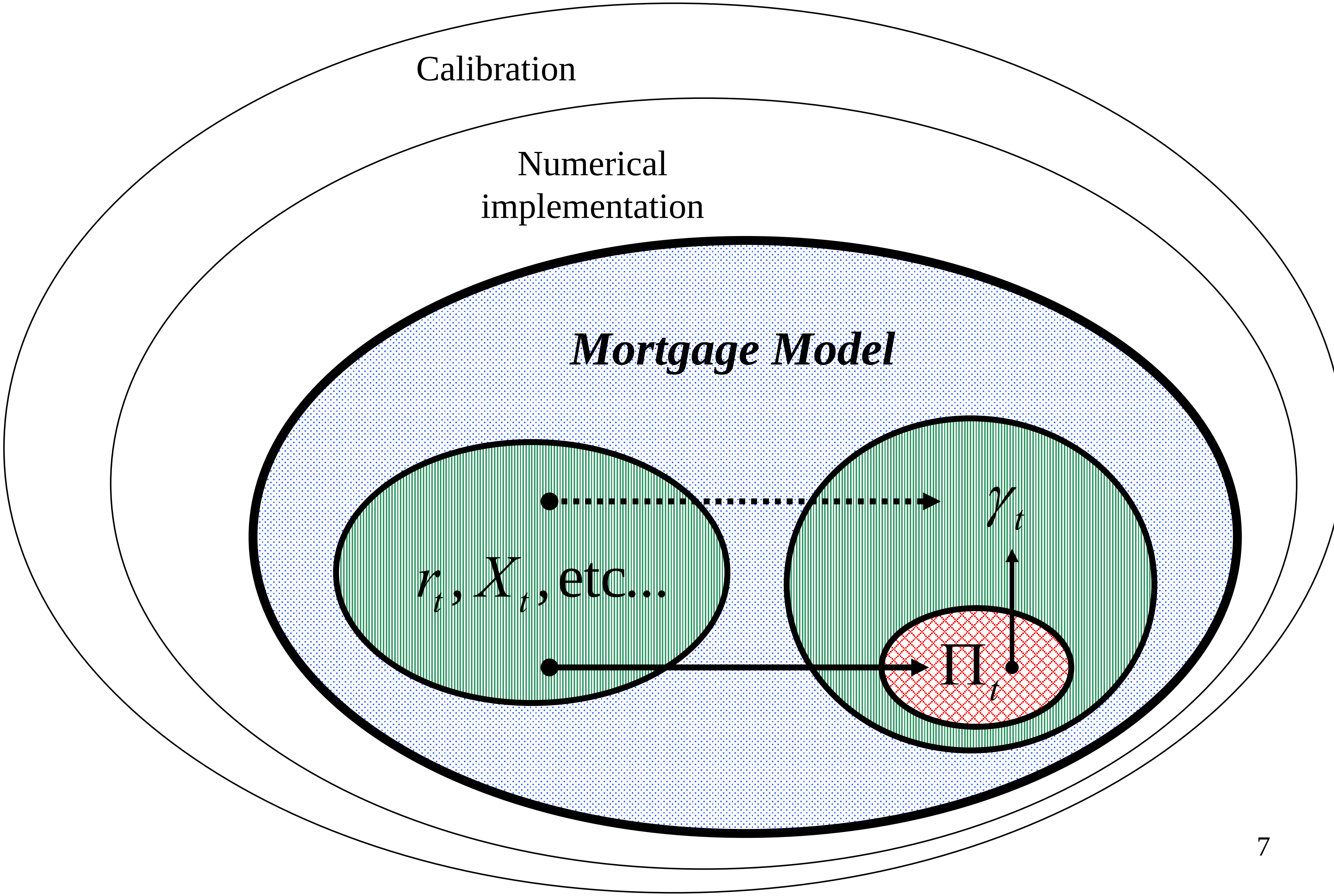
Empirical $\gamma^1(X_t^1)$ $\gamma^2(X_t^2)$ $\gamma^3(X_t^3)$ $\gamma^4(X_t^4)$

X_t^1 | X_t^2 | X_t^3 | X_t^4 time

Modeling $\gamma(\Pi_t^1)$ $\gamma(\Pi_t^2)$ $\gamma(\Pi_t^3)$ $\gamma(\Pi_t^4)$

not sensitive to time

Mortgage Modeling



Mortgage Model Classification

- Data, calibration
- Computational Method
- Interest Rate Model,

→ Not mortgage-specific. “Standard” problems

- Prepayment intensity function
- Additional predictors (house prices, media effect, etc)

→ Statistics

- **Refinancing incentive**

↓

Mortgage Model = Ref. Inc.

1. <u>Mort-Rate-Based</u>
2. <u>Option-Based</u>

Implied
Mortgage Rate Process

Let m^t be mortgage rate at time t :

$$M_t^{new} = P(0) \quad \Rightarrow \quad \mathbf{E}\left[\int_t^{t+T} P(s)\left(m^t - r_s\right) e^{-\int_t^s r_\theta + \gamma_\theta d\theta} ds \,\middle|\, \mathrm{F}_t \right] = 0$$

$$m^t = \frac{\mathbf{E}_t\left[\int_t^{t+T} r_s \, P(s\,; m^t)\, e^{-\int_t^s r_\theta + \gamma_\theta \, d\theta} ds \right]}{\mathbf{E}_t\left[\int_t^{t+T} P(s\,; m^t)\, e^{-\int_t^s r_\theta + \gamma_\theta \, d\theta} ds \right]} =: L_t^T (m^t \,|\{\gamma_s\}_{t<s<t+T})$$

The rate m^t implied by the prepayment process γ_t:

$$m^t = L_t^T (m^t \,|\{\gamma_s\}_{t<s<t+T})$$

Mortgage-Rate-Based Approaches

Examples: $\Pi_t = m^0 - m^t, \ \dfrac{m^0}{m^t}, \ \text{or} \ \dfrac{c(m^0)}{c(m^t)}$, i.e., $\boxed{\gamma_t = \gamma(m^0, m^t)}$

1. The process $\tilde{m}^t$ = the 10-year Treasury yield+const.

2. $m^t = L_t^T(m^t \,|\, \{\tilde{\gamma}_s\}_{t<s<t+T}) = \dfrac{\mathbf{E}[\int_t^{t+T} r_s P(s; m^t)\, e^{-\int_t^s [r_\theta + \gamma(\tilde{m}^t, \tilde{m}^\theta)] d\theta} ds]}{\mathbf{E}[\int_t^{t+T} P(s; m^t)\, e^{-\int_t^s [r_\theta + \gamma(\tilde{m}^t, \tilde{m}^\theta)] d\theta} ds]}$

in general $m^t \neq \tilde{m}^t$!

3. **Endogenous** mortgage rate $\{m^t\}_{t>0}$:

$$\boxed{m^t = L_t^T(m^t \,|\, \{\gamma(m^t, m^s)\}_{t<s<t+T})}$$

→ Pliska/Goncharov

→ MOATS

A Simple Example

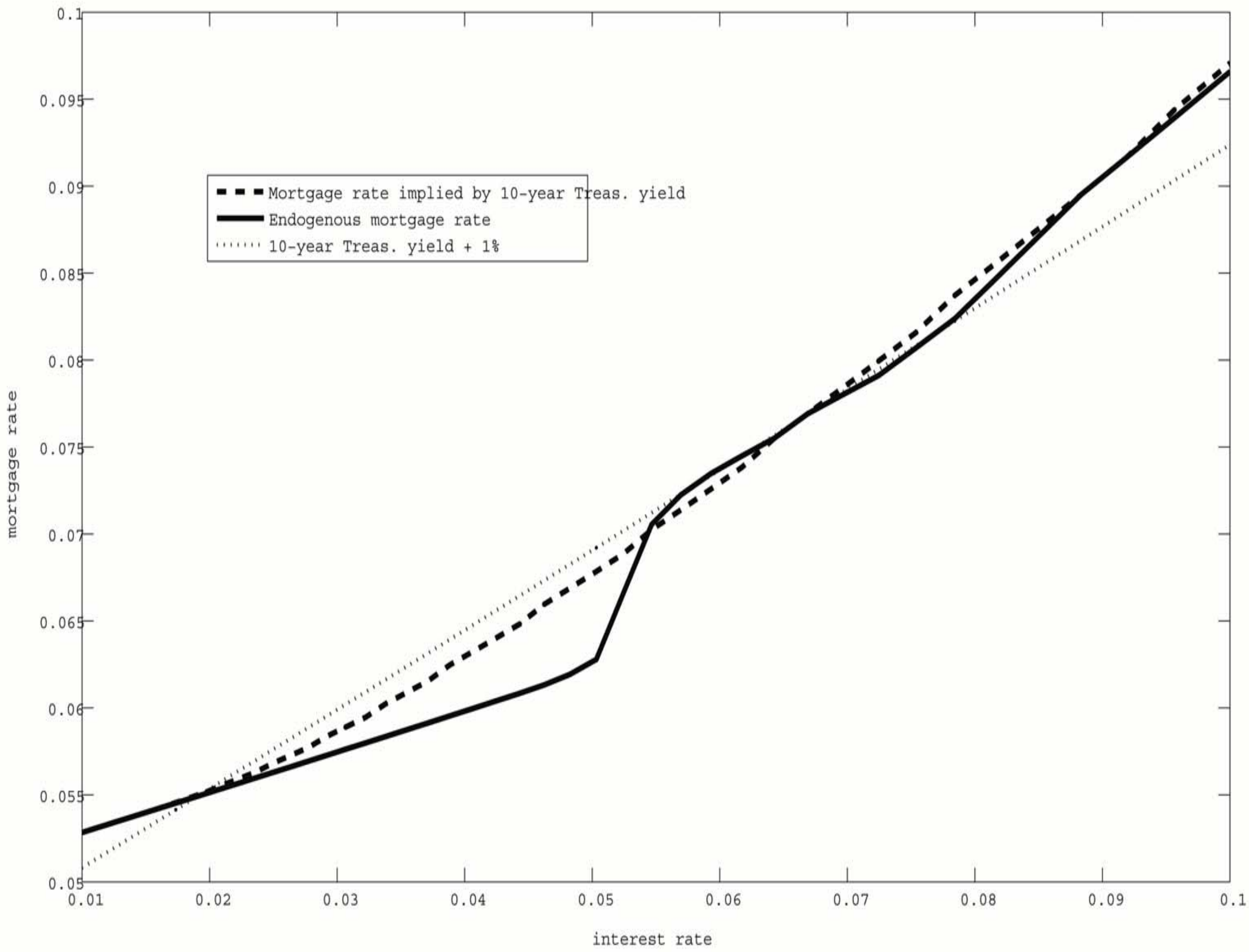
Consider a simple market which is completely described by a Markovian time-homogenous process r_t. Then

$$m^t = L_t^T(m^t \,\big|\, \{\gamma(m^t, m^s)\}_{t<s<t+T}) \square L_0^T(m^0 \,\big|\, \{\gamma(m^0, m^s)\}_{0<s<T})$$

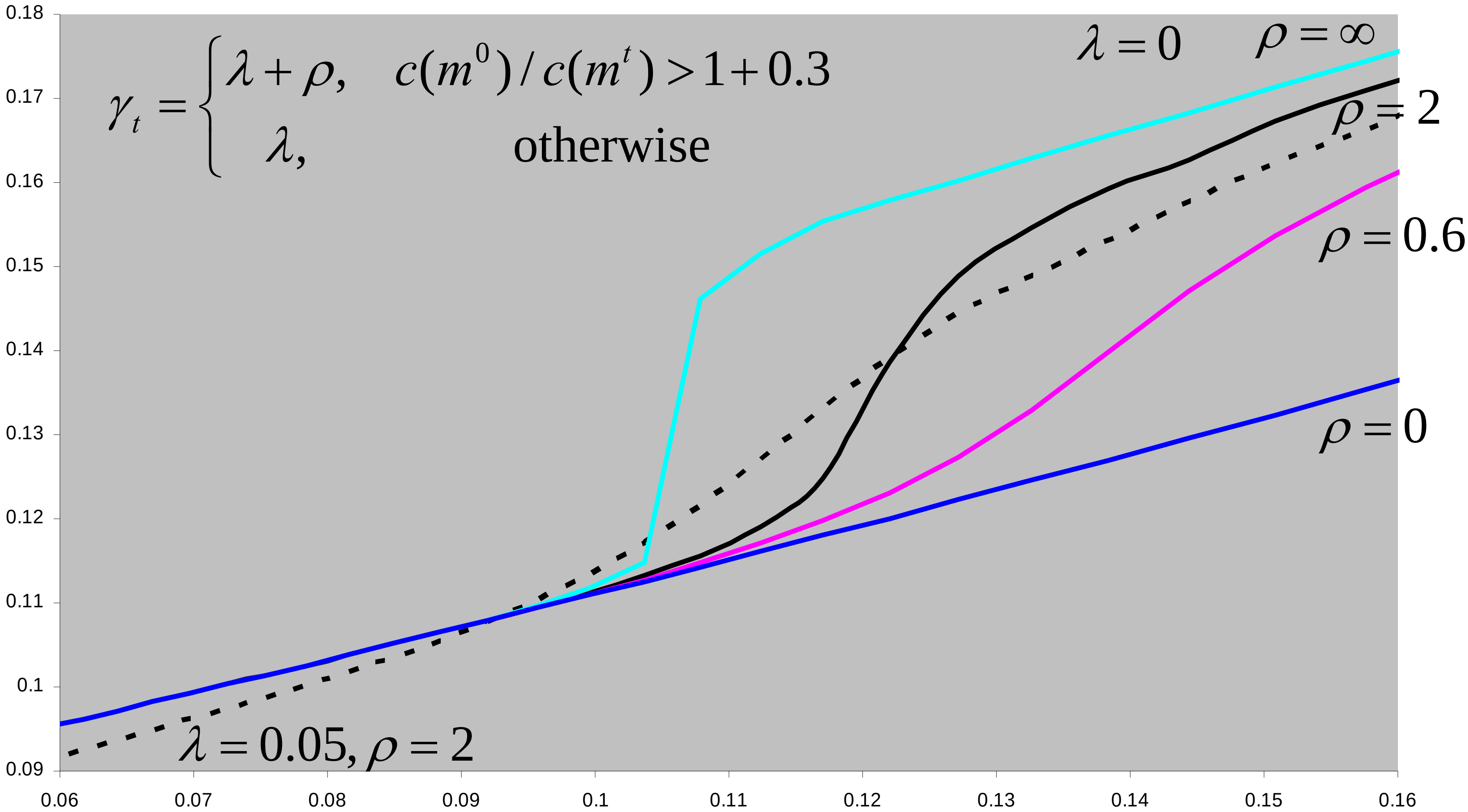
Thus $m^t = m(r_t),$ where

$$\boxed{m(r) = L(r, m(r) \,\big|\, \{m(\cdot)\})}$$

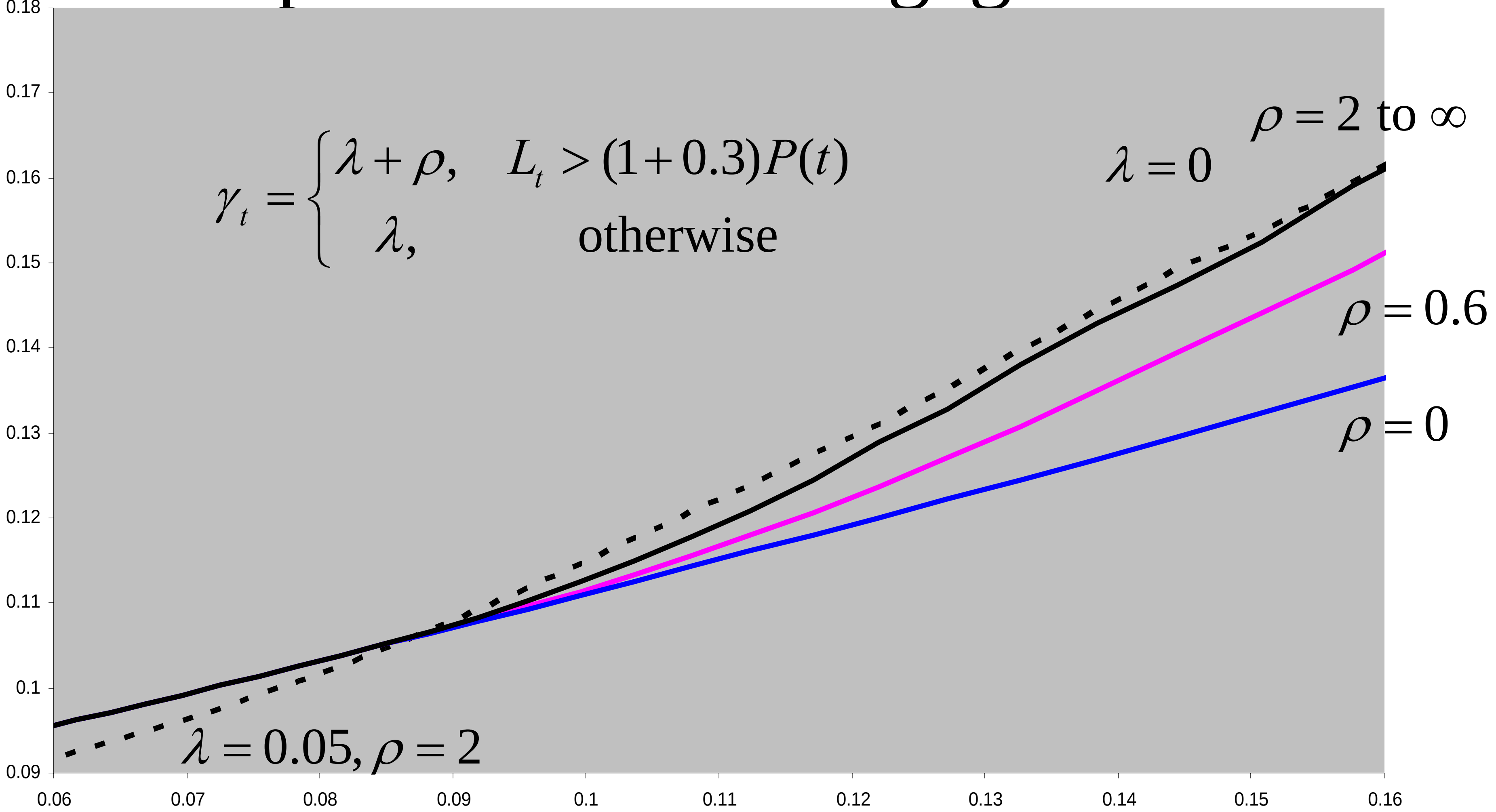
i.e., $$m(r) = \frac{\mathbf{E}[\int_0^T r_s\, P(s; m(r))\, e^{-\int_0^s r_\theta + \gamma(m(r), m(r_\theta))d\theta} ds \,\big|\, r_0 = r]}{\mathbf{E}[\int_0^T P(s; m(r))\, e^{-\int_0^s r_\theta + \gamma(m(r), m(r_\theta))d\theta} ds \,\big|\, r_0 = r]}$$


Mortgage rate implied by 10-year Treas. yield
Endogenous mortgage rate
10-year Treas. yield + 1%
mortgage rate
interest rate
0.05
0.055
0.06
0.065
0.07
0.075
0.08
0.085
0.09
0.095
0.1
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
0.1

MRB Mortgage Rate

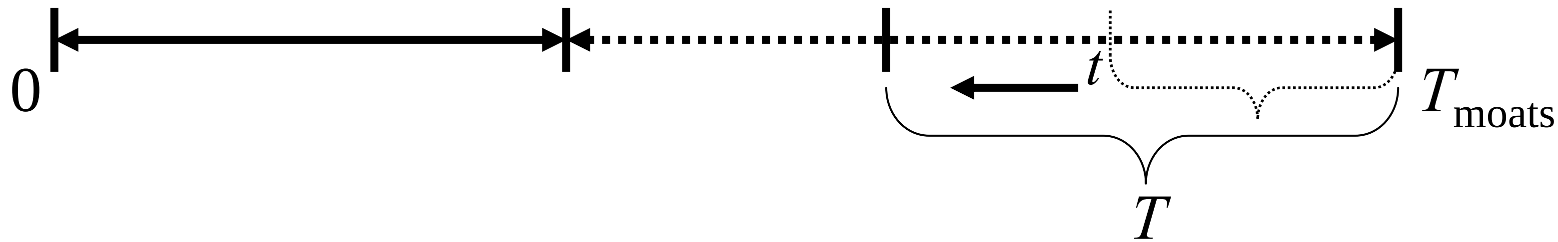


Option-Based Mortgage Rate

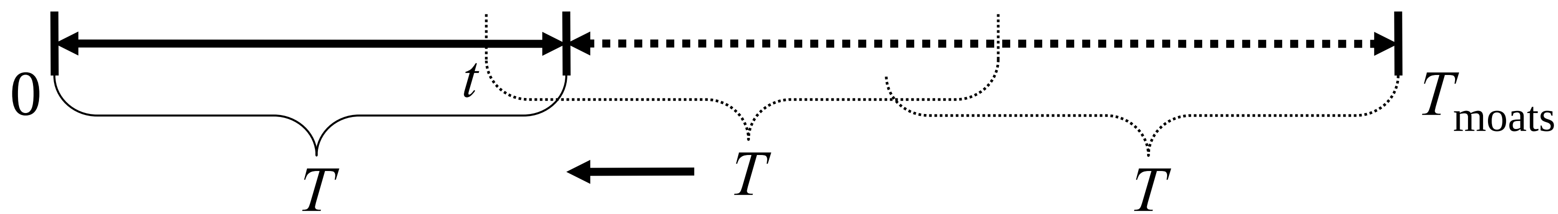


Citigroup's MOATS
(generalized)

1. For t from $T_{\text{moats}}-1$ to $T_{\text{moats}}-T$: $\quad m^t = L_t^{T_{\text{moats}}-t}\left(m^t \,\middle|\, \{\gamma(m^t, m^s)\}_{t<s<T_{\text{moats}}}\right)$

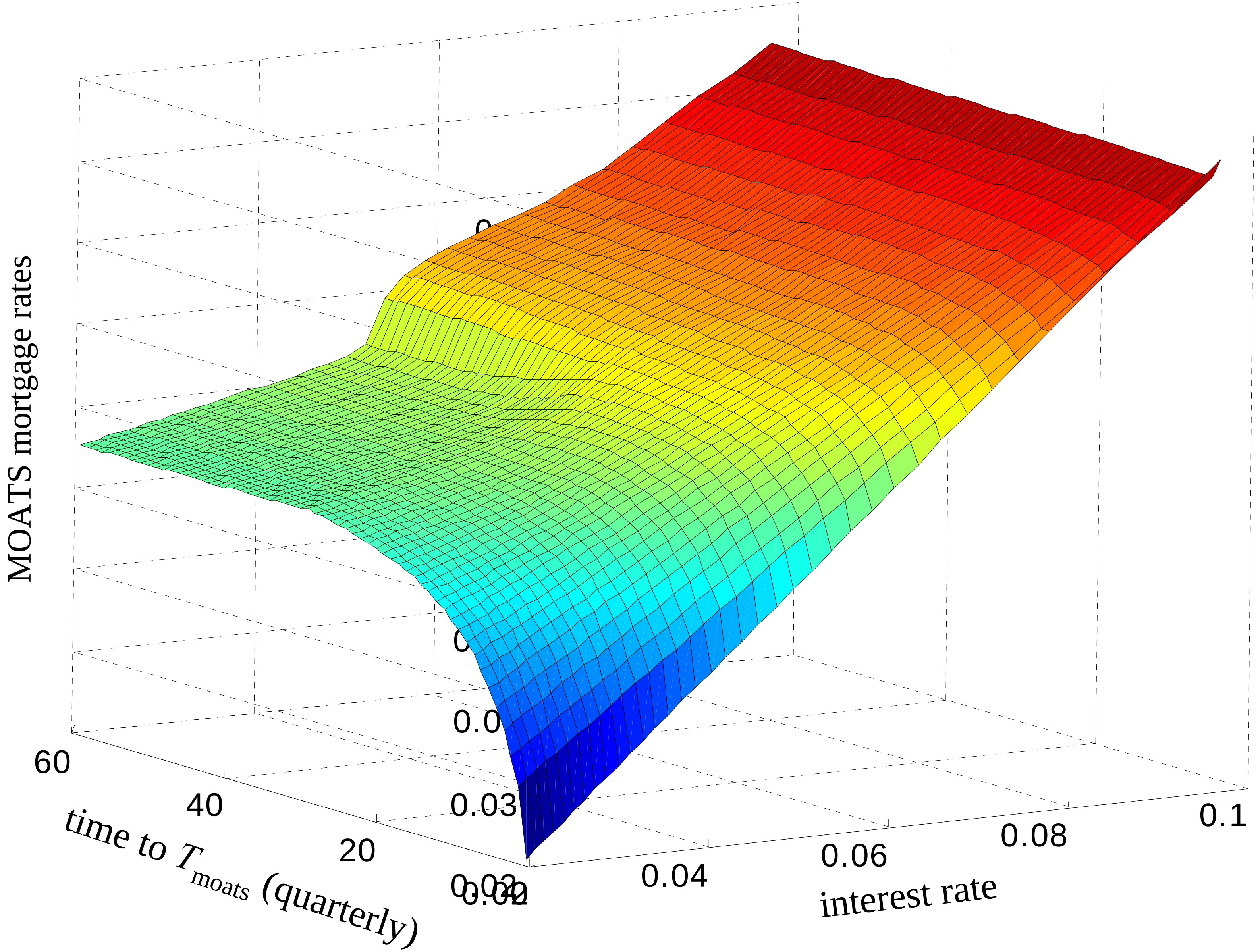


2. For t from $T_{\text{moats}}-T-1$ to 0: $\quad m^t = L_t^{T}\left(m^t \,\middle|\, \{\gamma(m^t, m^s)\}_{t<s<t+T}\right)$

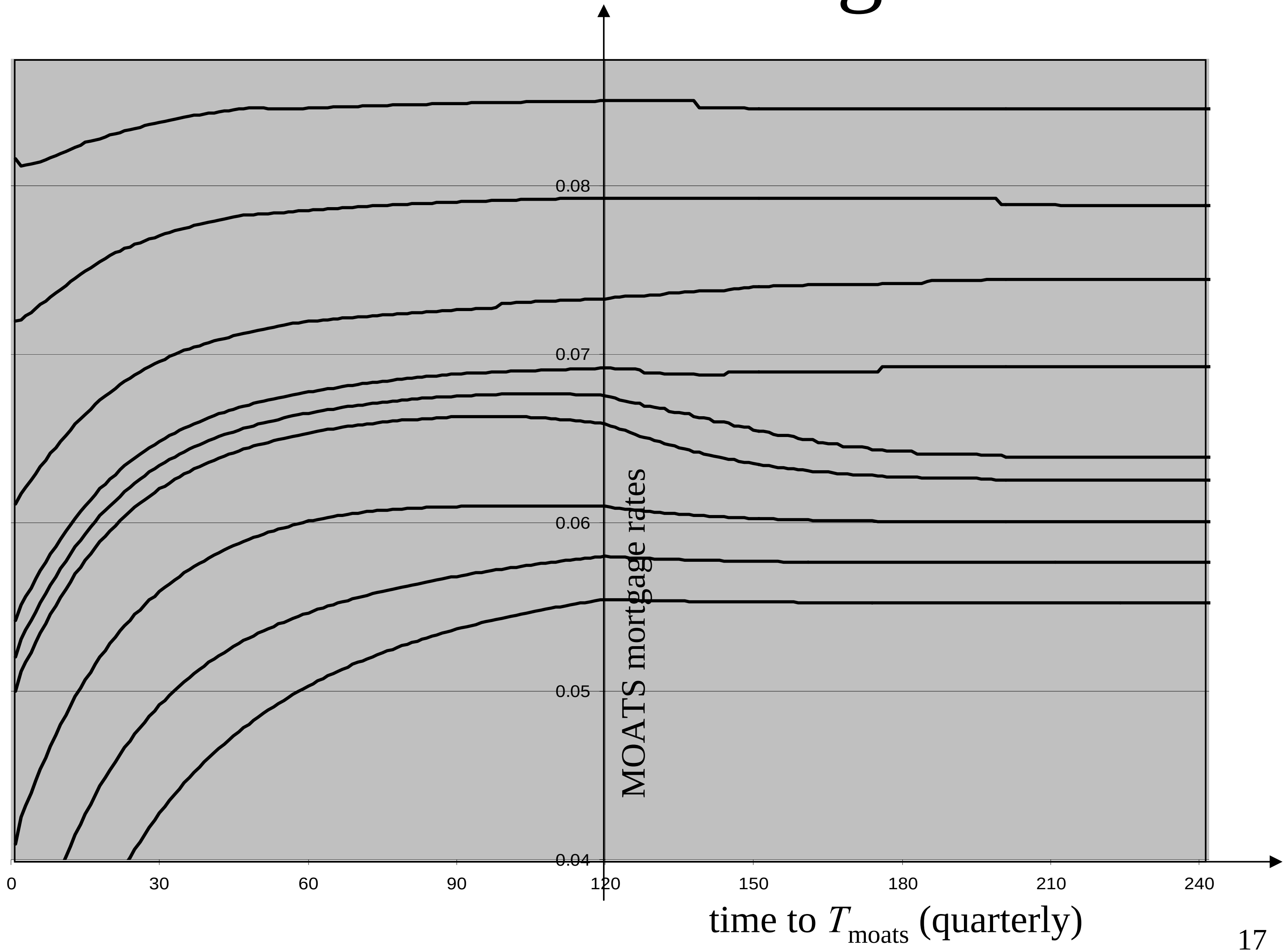


Citigroup:

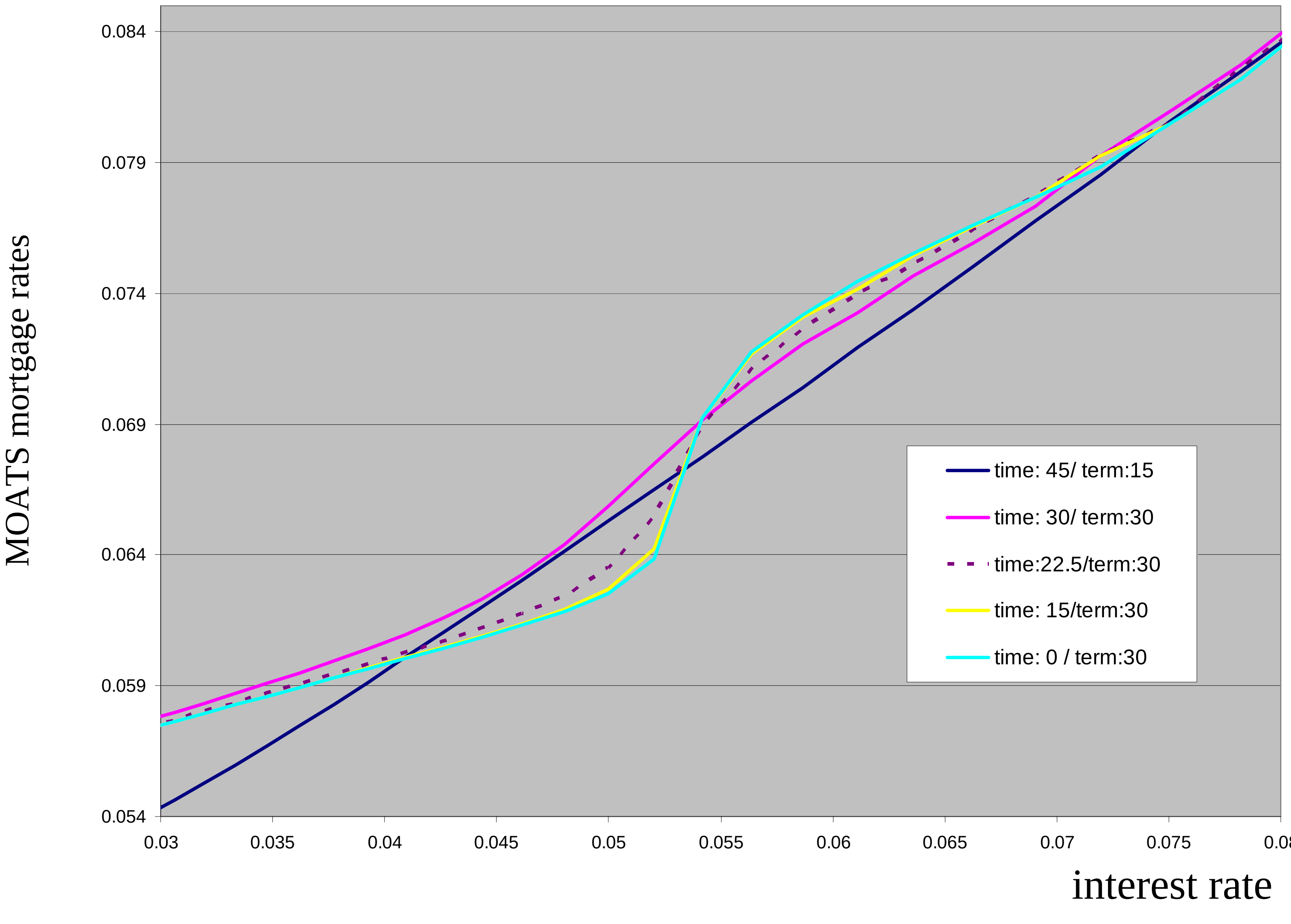
- T=360 (30 yr), T_{moats}=720 (60yr)
  comlpexity: (361*360/2+360*360)*N*I=194,580*N*I
- Interest only? One factor only?
- Historical dependence dropped, "calibrated" later…


MOATS mortgage rates
time to T_{moats} (quarterly)
interest rate
60
40
20
0.02
0.03
0.04
0.06
0.08
0.1

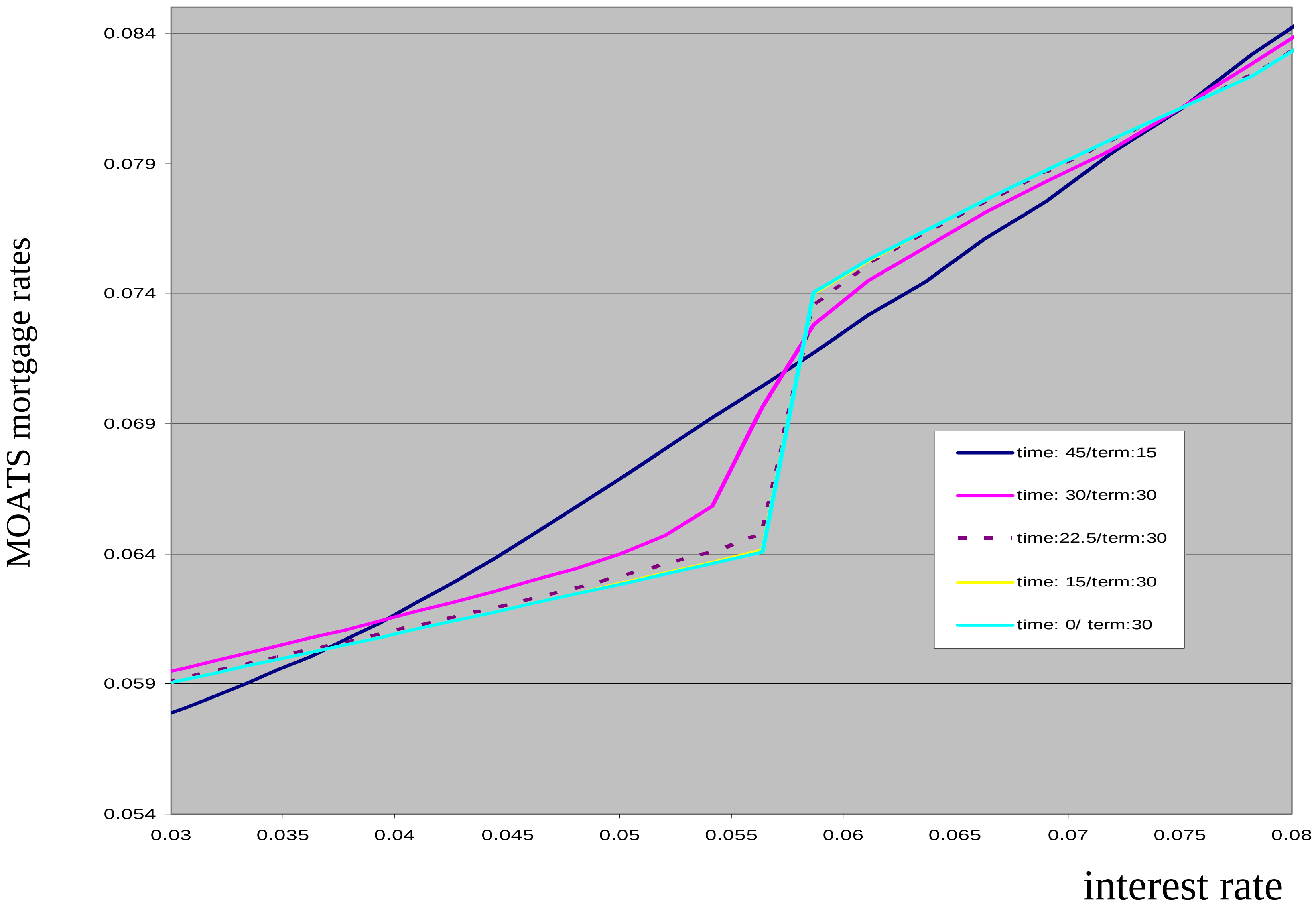
MOATS convergence



MOATS convergence



MOATS convergence (interest only)



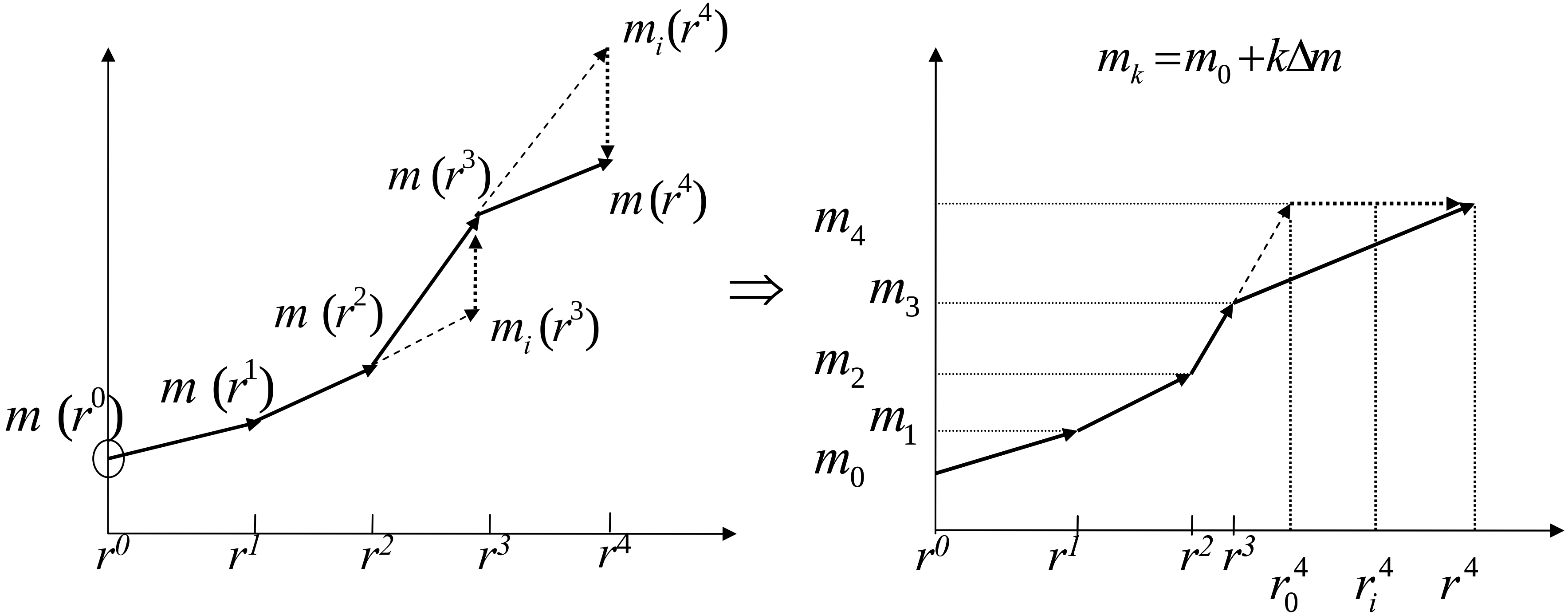
Endogenous Mort Rate Iteration

$$m(r) = \left. \frac{\mathbf{E}\left[\int_0^T r_s \, P(s, m(r)) e^{-\int_0^s r_\theta + \gamma(m(r), m(r_\theta)) d\theta} ds \,\middle|\, r_0 = x \right]}{\mathbf{E}\left[\int_0^T P(s, m(r)) e^{-\int_0^s r_\theta + \gamma(m(r), m(r_\theta)) d\theta} ds \,\middle|\, r_0 = x \right]} \right|_{x=r}$$

$$m_{i+1}(r) = L\left(x, m_i(r) \middle| \{ m_i(\cdot) \} \right) \Big|_{x=r}$$

- $m_{i+1}()$ requires estimation of $L()$ for "every" r?
- The result of $L()$-estimation is used at $x=r$ only, other values discarded?
- Curse of dimensionality with growth of r-dimension?

Mortgage Rate "Iterations"



Fix r then solve for $m(r)$:

$$m_{i+1}(r) = L(r, m_i(r) \,|\, \{m_i(\cdot)\})$$

the same…

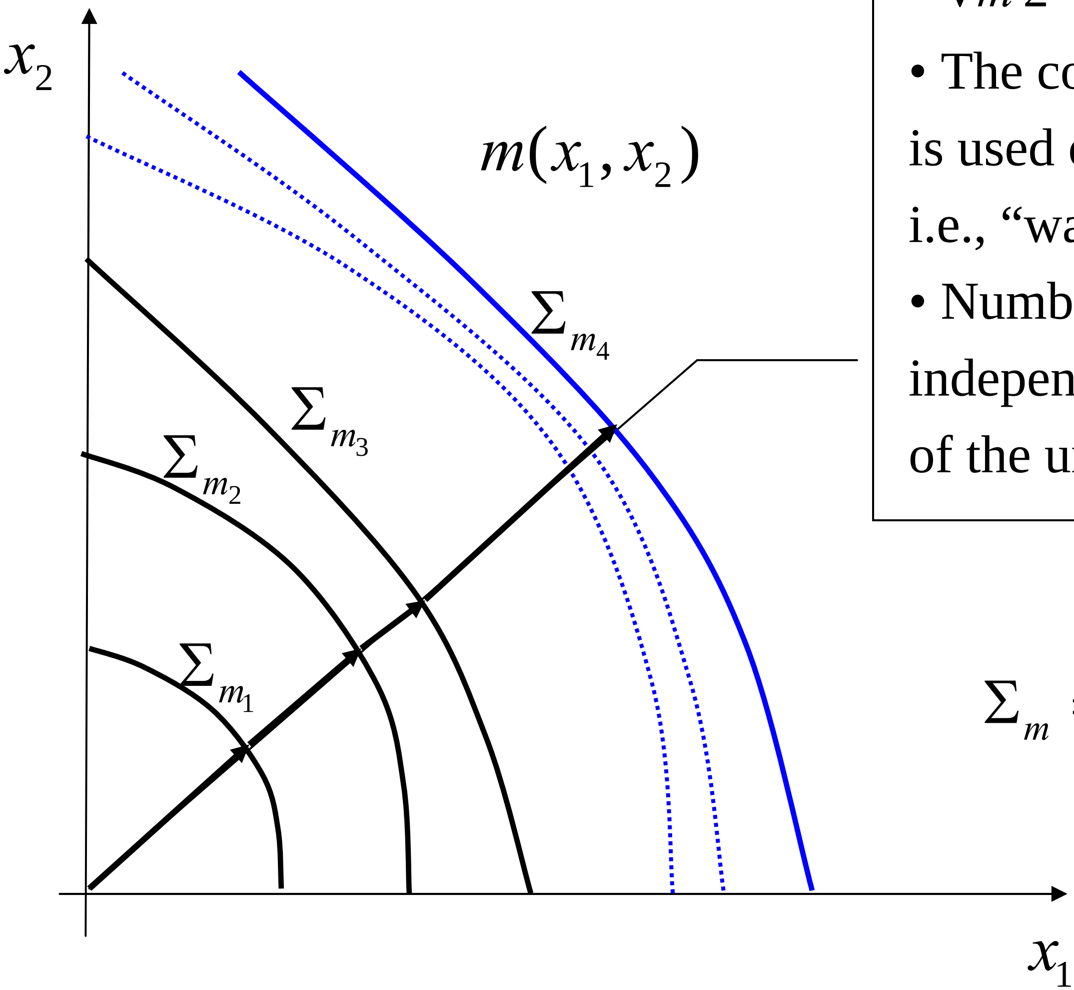
Fix m then solve for $r(m)$:

$$m = L(r_{i+1}, m \,|\, \{m_i(\cdot)\}) = L(r, m)$$

the same!

Refinancing region controlled

Computation with Level Sets



- No need for iterations if
  Vm £ "transaction costs"
- The conditional expectation in $L()$ is used on a hypersurface (level set), i.e., “waste of one dimension” only
- Number of $L()$-estimations is independent of the dimension/number of the underlying factors

$$\Sigma_m = \left\{ \vec{x} \,\middle|\, m = L\left(\vec{x}, m \,\middle|\, \left\{ m(\cdot) \right\} \right) \right\}$$

Conclusion

- Endogenous mortgage rate is defined
  - far from or implied by 10yr Treasury yield
  - accented nonlinear behavior
- MOATS
  - transparent definition → efficient implementation
  - convergence to MRB is shown
- A general ‘level set’ method is proposed
  - flexibility of implementation: [RQ]MC or PDE
  - reduces/eliminates the burden of iterations
  - complexity of the same order as the underlying problem
  - efficient and simple for the computation of implied mortgage rate given any prepayment model