UNIVERSITY OF ILLINOIS AT CHICAGO
Mechanical Engineering

IE 446
Solutions to Problem Set #2

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1. An estimator is unbiased if its expected value (mean) is equal to the parameter it estimates. For example, $E(S^2) = \sigma^2$. But the sample standard deviation S is not an unbiased estimator of the population standard deviation σ. Rather than explicitly calculate $E(S)$, consider:

$$\begin{aligned}
\mathrm{Var}(S) &= E(S^2) - E(S)^2 \\
E(S)^2 &= E(S^2) - \mathrm{Var}(S) \\
E(S) &= \sqrt{E(S^2) - \mathrm{Var}(S)} \\
E(S) &= \sqrt{\sigma^2 - \mathrm{Var}(S)} \\
E(S) &\neq \sigma
\end{aligned}$$

unless $\mathrm{Var}(S)$ is zero, which it is not. Note that we can further conclude that S underestimates σ.

2. The data for the problem are:

12.03 oz	12.01 oz
12.04	12.02
12.05	11.98
11.96	12.02
12.05	11.99

Thus we have the following statistics:

$$\begin{aligned}
\bar{x} &= 12.015 \\
S^2 &= .00091667 \\
S &= 0.30277 \\
n &= 10
\end{aligned}$$

(a) Since the variance is unknown, we use a t distribution for a confidence interval on the mean:

$$\bar{x} - t_{\frac{\alpha}{2},n-1}\frac{S}{\sqrt{n}} \leq \mu \leq \bar{x} + t_{\frac{\alpha}{2},n-1}\frac{S}{\sqrt{n}}$$

At $\alpha = 0.05$, the confidence interval is $[11.9933, 12.0367]$.

(b) We use a χ^2 distribution for a confidence interval on the variance of fill volume:

$$\frac{(n-1)S^2}{{\chi^2}_{\frac{\alpha}{2},n-1}} \leq \sigma^2 \leq \frac{(n-1)S^2}{{\chi^2}_{1-\frac{\alpha}{2},n-1}}$$

Calculating with $\alpha = 0.05$, the confidence interval is $[0.0004364, 0.0031]$.

(c) To test if the mean content exceeds 12.0 oz, use the hypotheses:

$$\begin{aligned}
H_0 &: \quad \mu = 12.0 \\
H_1 &: \quad \mu > 12.0
\end{aligned}$$

and the test statistic:

$$t_0 = \frac{\bar{x} - 12}{\frac{S}{\sqrt{n}}} = 1.5625$$

Since $t_{0.01,9} = 2.821 > t_0$, do *not* reject the null hypothesis that $\mu = 12.0$. The manufacturer cannot be sure that the mean net content exceeds 12.0 oz at $\alpha = 0.01$. In fact, since $t_{0.0763,9} = 1.5625$ (numerical solution with matlab), the P-value for this test is 0.0763, which means that H_0 will be rejected whenever the producer's risk α is at least 0.0763. (We must be willing to accept 7.6% of the lots where $\mu \leq 12$ in order to use this test to conclude that this particular lot has $\mu > 12.0$.)

3. This was not meant to be a trick question, but there is no closed-form exact solution. The test statistic for this hypothesis is

$$Z_0 = \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}} = \frac{(\bar{x} - \mu_0)\sqrt{n}}{\sigma}$$

which is distributed $N(0,1)$ *on the assumption that* H_0 *is true*. However, we assume that H_0 is false, and that the true mean is $\mu_1 = \mu_0 + \delta$. We wish to consider β, the probability that we will fail to reject H_0 in this case. If the true mean is μ_1, then Z_0 is distributed $N(\frac{\delta\sqrt{n}}{\sigma}, 1)$. Then

$$\begin{aligned} \beta &= P\left\{-Z_{\frac{\alpha}{2}} < Z_0 < Z_{\frac{\alpha}{2}}\right\} \\ &= \Phi\left(Z_{\frac{\alpha}{2}} - \frac{\delta\sqrt{n}}{\sigma}\right) - \Phi\left(-Z_{\frac{\alpha}{2}} - \frac{\delta\sqrt{n}}{\sigma}\right) \end{aligned}$$

where Φ is the cumulative normal distribution (which is tabulated in the appendix of Montgomery). Since α, δ, and σ can be treated as fixed parameters, β can be found as a function of n. The function cannot be inverted directly, but it is well-behaved (monotonically decreasing in n) so n can be found numerically as a function of β.

Note that in some cases it is reasonable to assume that $\Phi\left(-Z_{\frac{\alpha}{2}} - \frac{\delta\sqrt{n}}{\sigma}\right)$ is very small. (When is this reasonable?) If that is the case, then

$$\beta \approx \Phi\left(Z_{\frac{\alpha}{2}} - \frac{\delta\sqrt{n}}{\sigma}\right)$$

so

$$-Z_\beta \approx Z_{\frac{\alpha}{2}} - \frac{\delta\sqrt{n}}{\sigma}$$

which, after a little algebra, becomes

$$n \approx \frac{(Z_\beta + Z_{\frac{\alpha}{2}})^2\sigma^2}{\delta^2}$$

Note that this may underestimate n. The corresponding expression for a one-sided hypothesis *is* exact.

4. We wish to allocate $N = n_1 + n_2$ samples to test

$$\begin{aligned} H_0 &: \quad \mu_1 = \mu_2 \\ H_1 &: \quad \mu_1 \neq \mu_2 \end{aligned}$$

with the highest possible power (lowest β). When ${\sigma_1}^2$ and ${\sigma_2}^2$ are known, and the true mean is $\mu = \mu_0 + \delta$, we have:

$$\beta = \Phi(Z_{\frac{\alpha}{2}} - D) - \Phi(-Z_{\frac{\alpha}{2}} - D)$$

where

$$D = \frac{\delta}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

We can minimize D by minimizing $\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}$. Noting that

$$\frac{d}{dn_1}\left(\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}\right)=-\frac{\sigma_1^2}{n_1^2}+\frac{\sigma_2^2}{n_2^2}$$

we conclude that the most powerful test is achieved when $\frac{n_1}{n_2}=\frac{\sigma_1}{\sigma_2}$. This may mean we need to check the two closest integer values.

5. This is a Poisson distribution with $n = 100$ and 11 nonconformities (be careful: $\bar{x} = 0.11$, not 11). We test the hypothesis that the mean occurence rate of nonconformities is $\lambda_0 = 0.15$, using $\alpha = 0.01$. The test statistic is

$$Z_0=\frac{\bar{x}-\lambda_0}{\sqrt{\frac{\lambda_0}{n}}}$$

and the null hypothesis will be rejected if $|Z_0| > Z_{\frac{\alpha}{2}}$. Plugging in the numbers, we get

$$\begin{aligned}|Z_0| &= 1.0327\\ Z_{\frac{\alpha}{2}} &= 2.58\end{aligned}$$

so the null hypothesis is accepted. Note that the P-value is $\frac{\alpha}{2}\approx 0.15$, so this is a fairly resounding acceptance.