UNIVERSITY OF ILLINOIS AT CHICAGO
Mechanical Engineering

IE 446
Solutions to Problem Set #3

Michael J. Scott Spring 2000

1. We use the following normally distributed data:

Technician 1	1.45	1.37	1.21	1.54	1.48	1.29	1.34	
Technician 2	1.54	1.41	1.56	1.37	1.20	1.31	1.27	1.35

Thus $n_1 = 7$, $n_2 = 8$. We can calculate the sample means and variances for each set of data:

$$\begin{aligned} \bar{x}_1 &= 1.382857 \\ \bar{x}_2 &= 1.376250 \\ S_1^2 &= 0.013190 \\ S_2^2 &= 0.015598 \end{aligned}$$

I used matlab and the functions nanmean and nanstd to get these numbers. For example:

```
>> m1 = nanmean([1.45  1.37 1.21 1.54 1.48 1.29 1.34])
m1 =
   1.38285714285714

>> s1 = nanstd([1.45  1.37 1.21 1.54 1.48 1.29 1.34])^2
s1 =
   0.01319047619048
```

(a) Since we assume variances are equal but unknown, we will use a t distribution with $n_1 + n_2 - 2$ degrees of freedom and the pooled variance

$$S_p^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2} = 0.014487$$

The two-sided 95% confidence interval ($\frac{\alpha}{2} = 0.025$) is given by

$$\bar{x}_1 - \bar{x}_2 - t_{\frac{\alpha}{2},n_1+n_2-2} S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \leq \mu_1 - \mu_2 \leq \bar{x}_1 - \bar{x}_2 + t_{\frac{\alpha}{2},n_1+n_2-2} S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

We must look up or calculate $t_{0.025,13} = 2.16$. It's in the table, or use matlab:

```
>> -tinv(0.025,13)
ans =
   2.16036865646279
```

Applying all the numbers, get:

$$-0.127969 \leq \mu_1 - \mu_2 \leq 0.141183$$

(b) The confidence interval on the ratio of variances uses an F distribution:

$$\frac{S_1^2}{S_2^2} F_{1-\frac{\alpha}{2},n_2-1,n_1-1} \leq \frac{\sigma_1^2}{\sigma_2^2} \leq \frac{S_1^2}{S_2^2} F_{\frac{\alpha}{2},n_2-1,n_1-1}$$

The table says $F_{0.025,7,6} = 5.7$. Matlab says:

```
>> finv(0.975,7,6)
ans =
   5.69547047368316
```

(Note that the conventions are different.) Following the note at the bottom of the table:

$$F_{0.975,7,6} = \frac{1}{F_{0.025,6,7}} = \frac{1}{5.12} = 0.195312$$

Matlab says:

```
>> finv(0.025,7,6)
ans =
   0.19536604962876
```

which is presumably more accurate. In any event, we can calculate the confidence interval:

$$\frac{0.0131905}{0.0155982}0.195366 = 0.165209 \leq \frac{\sigma_1^2}{\sigma_2^2} \leq \frac{0.0131905}{0.0155982}5.695470 = 4.816318$$

Since the confidence interval includes 1, the assumption that the two variances were equal was reasonable (to a confidence level of 95%).

(c) Need a χ^2 distribution for a confidence interval on σ_2^2 ($n = 8$ for technician 2):

$$\begin{aligned}
\frac{(n-1)S_2^2}{\chi^2_{\frac{\alpha}{2},n-1}} &\leq \sigma_2^2 \leq \frac{(n-1)S_2^2}{\chi^2_{1-\frac{\alpha}{2},n-1}} \\
\frac{(7)0.015598}{16.012764} &\leq \sigma_2^2 \leq \frac{(7)0.015598}{1.689869} \\
0.006819 &\leq \sigma_2^2 \leq 0.064613
\end{aligned}$$

2. (a) Test on the mean, $\sigma = 2$ is known, $n = 8$ sampled, $\bar{x} = 127$ is calculated. The 95% confidence interval uses $Z_{0.05} = 1.644854$ (use `norminv(0.95,0,1)` in matlab, or estimate from the table), and is:

$$\mu \geq \bar{x} - Z_{0.05}\frac{\sigma}{\sqrt{n}} = 127 - 1.644854\frac{2}{\sqrt{8}} = 125.83691$$

(b) Since the lower confidence interval has a lower bound of 125.83691, we can conclude immediately that we must reject the null hypothesis that $\mu = 125$ and accept the alternative hypothesis that $\mu > 125$. (You can do the algebra if you like.) The P-value is the smallest value of α for which we reject the null hypothesis. Now we do need the test statistic:

$$Z_0 = \frac{(\bar{x} - \mu)\sqrt{n}}{\sigma} = \frac{(127 - 125)\sqrt{8}}{2} = \sqrt{8} = 2.828427$$

We want to find α such that $Z_\alpha = 2.828427$, which means that we need to find $\Phi(2.828427)$. From the table, we see that $\Phi(2.82) = 0.99760$ and $\Phi(2.83) = 0.99767$, so we can conclude that the P-value is between 0.00233 and 0.00240. We can also use matlab:

```
>> 1 - normcdf(sqrt(8),0,1)
ans =
   0.00233886749052
```

A P-value of about 0.00234, or 2.34%, means that we are able to conclude that $\mu > 125$ (*i.e.*, that the alternative hypothesis is correct) if we are prepared to have that conclusion be in error 2.34% (or more) of the time.

3. This is a test on the difference in means, variances known. We'd better assume normal distributions (and you ought to say that). Since $\sigma_1 = \sigma_2 = \sigma$ and $n_1 = n_2 = n$, the test statistic is

$$Z_0 = \frac{\bar{x_2} - \bar{x_1} - 5}{\sqrt{\frac{2\sigma^2}{n}}} = \frac{176.8 - 170.8 - 5}{\sqrt{\frac{2(6.76)}{16}}} = 1.087857$$

Since $Z_{0.05} = 1.644854$, we cannot reject the null hypothesis that the difference in means is equal to five (in other words, we cannot accept the alternative hypothesis that the difference in means is greater than five). Thus the manufacturer should stick with bottle 1. The P-value is $1 - \Phi(1.087857) = 0.138329$ (table or matlab).

4. Recall that we found both approximate and exact solutions. Since we have the power of matlab at our disposal, we will use the exact solution:

$$\beta = \Phi\left(Z_{\frac{\alpha}{2}} - \frac{\delta\sqrt{n}}{\sigma}\right) - \Phi\left(-Z_{\frac{\alpha}{2}} - \frac{\delta\sqrt{n}}{\sigma}\right)$$

For all problems, $\delta = 1.5$ and $\sigma = \sqrt{5}$. We can write a little matlab script to find β as a function of n. I call this file findbeta.m:

```
function out = findbeta(delta,n,sigmasquared,alpha)
d = delta*sqrt(n/sigmasquared);
za = norminv(1-alpha/2);
out = normcdf(za-d,0,1) - normcdf(-za-d,0,1);
```

I can use this to find β for any set of input parameters, for instance:

```
>> findbeta(1.5,10,5,0.05)
ans =
    0.4359
```

says that if $n = 10$, then $\beta = 0.4359$ (much too large). By using a numerical solver we can find n for a particular β, as in this example where $\beta = 0.05$:

```
>> fzero( 'findbeta(1.5,x,5,0.05)-0.05',30)
ans =
   28.8771
```

(a) Since $\beta = 0.0560$ for $n = 28$, and $\beta = 0.0492$ for $n = 29$, we conclude that $n = 29$ is required to achieve $\beta = 0.05$ when $\alpha = 0.05$.

(b) $n = 40$ is required to achieve $\beta = 0.05$ when $\alpha = 0.01$.

(c) $n = 41$ is required to achieve $\beta = 0.01$ when $\alpha = 0.05$.

(d) $n = 54$ is required to achieve $\beta = 0.01$ when $\alpha = 0.01$.

(e) Draw your own general conclusions; no reasonable ideas refused.