# Some Notes on Necessary and Sufficient Conditions

Craig Fox

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## 1 Two types of conditions.

One approach to providing an analysis of our talk of relationships between different events (e.g., causes and effects) is through an examination of various *conditions* under which a given event does or does not occur. Let us highlight two kinds of types of conditions. The first is called a **necessary condition**. We'll call the event in which we're interested $\mathcal{E}$. (This event $\mathcal{E}$ could be anything: a car starting, a match lighting, a team winning a game,...) If something (let's call it $\mathcal{N}$) *must* happen in order for $\mathcal{E}$ to occur, then whatever that is ($\mathcal{N}$) is a **necesary condition** of $\mathcal{E}$'s happening. Another way of saying this is that if $\mathcal{N}$ does not occur, then $\mathcal{E}$ cannot either.

The second type of condition is known as a **sufficient condition**. If something (let's call it $\mathcal{S}$) always leads $\mathcal{E}$ to occur, then $\mathcal{S}$ is a sufficient condition for $\mathcal{E}$'s ocurring. Another way of saying this is that whenever $\mathcal{S}$ occurs, then so does $\mathcal{E}$.

### 1.1 Definitions.

Let us call the event in which we're interested: $\mathcal{E}$.

A **necessary condition** $\mathcal{N}$ is something that must occur if $\mathcal{E}$ is to occur.

A **sufficient condition** $\mathcal{S}$ is something such that if it occurs, then so will $\mathcal{E}$.

### 1.2 Examples.

Let's consider, as our event $\mathcal{E}$, "winning the lottery." Can we come up with something that must occur whenever someone wins the lottery? An obvious choice is that one must have purchased a ticket for that particular game. Let us call "purchasing a ticket for that game" $\mathcal{P}$. Is $\mathcal{P}$ a necessary or a sufficient condition for $\mathcal{E}$? Since $\mathcal{P}$ *must* occur if $\mathcal{E}$ is to occur, then $\mathcal{P}$ is properly classified as a necessary condition for $\mathcal{E}$. Again: a necessary condition for winning the lottery is buying a ticket for that particular game. Is $\mathcal{P}$ also a sufficient condition for $\mathcal{E}$? In order for that to be the case, whenever $\mathcal{P}$ would occur, then so would $\mathcal{E}$. Since it's not the case that purchasing a ticket ensures that one will win the lottery (i.e. one can—and often *does*—purchase a ticket and *not* win), we know that $\mathcal{P}$ is not a sufficient condition of $\mathcal{E}$'s happening.

As another example, let's consider "passing Phil 102" as our event $\mathcal{E}$. As a careful and attentive reader of the course syllabus, you've noted that each student must take the final exam in order to be able to receive a passing grade in the course. (Any student who does not take the final cannot pass the course.) So let's call "taking the final exam" $\mathcal{T}$. What can we say of the relationship between $\mathcal{E}$ and $\mathcal{T}$? One thing that we can say is that if one doesn't take the final exam, then one will not pass the course. Here we can consider $\sim \mathcal{T}$ as a sufficient condition for $\sim \mathcal{E}$. (The notation '$\sim \mathcal{E}$' means simply that $\mathcal{E}$ does not occur.) Another thing that we can say, however, is that if $\mathcal{E}$ does happen, then $\mathcal{T}$ must have occured. So $\mathcal{T}$ is a necessary condition for $\mathcal{E}$.

In order to see how well you've understood the material this far, here are a couple of questions:

**(i)** What is the relationship (noted in the last paragraph) *between* necessary/sufficient conditions *and* events occurring/not occurring?

**(ii)** Suppose we give a *definition* of what it means for some event $\mathcal{E}$ to occur. What's important about a definition, in this sense, is that it would tell us when $\mathcal{E}$ would occur as well as when it would *not*. Which kind of conditions would this definition give us? (Hint: consider $\mathcal{E}$="winning the lottery". What might a definition of "winning the lottery" be like? How would we classify the conditions under which the definition is satisfied or isn't?)

## 2 Regimentation of Conditions.

Explaining why something is a necessary or a sufficient condition leads us relatively easily towards a way of providing a regimentation that captures the logical relationship between the events under consideration. Our task with the regimentation of necessary and sufficient conditions is the same as it is elsewhere in logic: we're aiming to capture, with our symbolic version, the truth-conditions of the original English sentence. So, if I say that I must purchase a ticket in order for me to win the lottery, then that means that any time I've won the lottery, I must have purchased a ticket. In other words, *if* I've won the lottery, *then* I've purchased a ticket. Hence, the regimentation here would be, as might be expected at this point: $(W \supset P)$.

Similarly, if I say that "not taking the final exam" is sufficient for "not passing Phil 102," I mean that *if* I don't take the final exam, *then* I won't pass the course. Hence the appropriate regimentation would be: $(\sim T \supset \sim P)$.

If we generalize upon these examples, we arrive at the following rule of thumb: A sufficient condition for something occuring will be regimented as the antecedent in a conditional. A necessary condition of something ocurring will be regimented as the consequent in a conditional. (A mnemonic device for remembering this relationship is the following: remember **SUN**, which comes from **S**ufficient condition $\supset$ **N**ecessary condition. The horseshoe becomes the 'U'.)

Also note the following variation. I can express the idea that purchasing a ticket is necessary for winning the lottery equally well by saying that I cannot win the lottery unless I've purchased a ticket. This suggests that I can regiment my necessary condition with: $(\sim W \vee P)$. In fact, this formula is logically equivalent to (means the same as) the conditional regimentation.

Sometimes there may be multiple conditions present in a given situation. For example, consider: "Striking a match in the presence of oxygen leads to fire." So our event here is "fire." We have two sufficient conditions that cause the fire: "striking a match," and "being in the presence of oxygen." Neither one, by itself, is sufficient. We can express the logical relationship, then, as the following: $(S \cdot O) \supset F$. The key to this regimentation is that *our 'simples' are the different conditions, which we can individuate*—as opposed to the 'simples' being just complete sentences. A negation represents a condition's absence.

We can go further with this last example, as well. $O$ can also be considered as a necessary condition for fire, on its own, as there cannot be fire without oxygen present. Hence we can assert: $(F \supset O)$.

Also of note in this example is that the conditions are, we might say, **jointly sufficient**. That is, it is the two conditions taken as a unit (a conjunction) that is sufficient—each condition *on its own* is not a sufficient condition. Compare the situation with the following example, however. Take, as $\mathcal{E}$, "winning the lottery." We might give as a necessary condition of $\mathcal{E}$ that one must have purchased a ticket, *and* that one must have picked the correct numbers. Hence we could regiment the logical situation as: $(W \supset (P \cdot H))$. In contrast with the previous example, each of $P$ and $H$—on its own—is a necessary condition of $\mathcal{E}$.

## 3 Exercises.

**(i)** Classify any of the conditions in each of the following situations as necessary ($\mathcal{N}$) or sufficient ($\mathcal{S}$), as well as what they're conditions *of* ($\mathcal{E}$). (Leave blank any type of condition that does not occur.) **(ii)** Then provide a regimentation that captures the logical relationship between the events under consideration. If multiple classifications and regimentations are possible, list alternates as well.

PLEASE WRITE YOUR ANSWERS ON A SEPARATE SHEET.

(1) "Arguing a judgment call will get you thrown out of the game."
$\mathcal{E}$=
$\mathcal{N}$=
$\mathcal{S}$=
Regimentation=

(2) "You will get your license suspended when you drive while intoxicated."
$\mathcal{E}$=
$\mathcal{N}$=
$\mathcal{S}$=
Regimentation=

(3) "You can't have a fire without oxygen."
$\mathcal{E}$=
$\mathcal{N}$=
$\mathcal{S}$=
Regimentation=

(4) "You gotta be in it to win it."
$\mathcal{E}$=
$\mathcal{N}$=
$\mathcal{S}$=
Regimentation=

(5) "In order to be able to ride, one must have one red or two blue tickets."
$\mathcal{E}$=
$\mathcal{N}$=
$\mathcal{S}$=
Regimentation=

(6) "You must pay a $ 30 fine, unless you appeal and your appeal is successful."
$\mathcal{E}$=
$\mathcal{N}$=
$\mathcal{S}$=
Regimentation=

(7) "Registering without permission will invalidate your schedule."
$\mathcal{E}$=
$\mathcal{N}$=
$\mathcal{S}$=
Regimentation=

(8) "You'll pass only if you study, although if this all comes naturally then you'll pass too."
$\mathcal{E}$=
$\mathcal{N}$=
$\mathcal{S}$=
Regimentation=