4 The Arbitrage Theorem

4.1 Definition

- To arbitrage is to take simultaneous positions in different assets in a way that guarantees a riskless profit higher than the return on the riskless asset.

- Types of arbitrage opportunities
  - **First kind.** The opportunity to make investments with no current net commitment that have a positive profit.

    For example, short-sell one asset and we use the proceeds to buy another in such a way as to make the portfolio riskless.
  - **Second kind.** The opportunity to make investments with a negative net commitment today (i.e., money comes to you and you have some left over after you make your asset purchases) and that yield non-negative profit.
- **Fair price** (correct price). Prices are fair (or correct) if and only if there are no arbitrage opportunities.

4.2 Notation

- **Asset prices.** Individual security prices are denoted by $S_i(t)$.

  The set of prices for all (relevant) assets is the vector

$$S_t = \begin{bmatrix} S_1(t) \\ S_2(t) \\ \vdots \\ S_N(t) \end{bmatrix}$$

- **States of the world**

$$W = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_K \end{bmatrix}$$

  - Where each w_i represents a distinct outcome that may occur.
  - The states are mutually exclusive
  - At least one of them is guaranteed to occur.

- **Returns and payoffs**
  - Total payments in one period made on security i when state j prevails is d_{ij}.
  - Components of d_{ij}
    * Capital gain – change in the value of the asset. Can be negative
    * Dividends or coupon payments
  - Payment matrix. For the N assets, we have the payment matrix

$$D = \begin{bmatrix} d_{11} & \cdots & d_{1K} \\ \vdots & \ddots & \vdots \\ d_{N1} & \cdots & d_{NK} \end{bmatrix}$$

    * Rows represent payment from a particular security in different states of the world.
    * Columns represent payments from different assets in a particular state.

- **Portfolio**
  - A particular combination of the assets in question
  - Represented by the vector

$$\theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_N \end{bmatrix}$$

    where θ_i is the amount/number of asset in the portfolio.
  - Negative θ_i represents a short position in the ith asset.

4.3 Example

1. **Three assets**

   - Asset 1: riskless with initial value $B(t)$ and gross return of $(1 + r\Delta)$, where r is the riskless rate and Δ is the length of the period.
   - Asset 2: underlying risky asset with value $A(t)$. Can assume only two possible values.
   - Asset 3: a derivative asset with value $C(t)$. This asset expires next period. Because the underlying asset has only two possible values, $C(t)$ has only two possible values.

- The asset vector is

$$S_t = \begin{bmatrix} B(t) \\ A(t) \\ C(t) \end{bmatrix}$$

- The payment matrix is

$$D_t = \begin{bmatrix} (1 + r\Delta)B(t) & (1 + r\Delta)B(t) \\ A_1(t + \Delta) & A_2(t + \Delta) \\ C_1(t + \Delta) & C_2(t + \Delta) \end{bmatrix}$$

- Simplification
  - We are interested in the relative values of B, A and C.
  - We can choose any convenient absolute size of the portfolio.
  - It is convenient to choose $B(t) = 1$.
  - Similarly, the length of one period is arbitrary, so for convenience choose $\Delta = 1$.
  - Therefore, we have

$$S_t = \begin{bmatrix} 1 \\ A(t) \\ C(t) \end{bmatrix}$$

$$D_t = \begin{bmatrix} (1+r) & (1+r) \\ A_1(t+1) & A_2(t+1) \\ C_1(t+1) & C_2(t+1) \end{bmatrix}$$

2. **Arbitrage theorem**

- **Theorem:** There are no arbitrage possibilities if and only if **positive** constants ψ_1 and ψ_2 exist such that

$$\begin{bmatrix} 1 \\ A(t) \\ C(t) \end{bmatrix} = \begin{bmatrix} (1+r) & (1+r) \\ A_1(t+1) & A_2(t+1) \\ C_1(t+1) & C_2(t+1) \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \tag{1}$$

- Some questions:

(a) How does one prove the theorem?

(b) What is the intuition for the theorem? Why does existence of (ψ_1, ψ_2) guarantee absence of arbitrage?

(c) What interpretation can we give ψ_1 and ψ_2?

(d) How is the theorem relevant to asset pricing?

For now, we ignore (a) and (b).

3. **Interpretation of (ψ_1, ψ_2).**

- Suppose $A_1(t+1) = 1$ and $A_2(t+1) = 0$
- Then the second row of (1) gives

$$A(t) = 1 \times \psi_1 + 0 \times \psi_2 = \psi_1$$

  $\Rightarrow$ Investors are willing to pay ψ_1 units of account today for an "insurance policy" that return 1 unit of account in state 1 and 0 units of account in state 2.

- Similarly, supposing that $A_1(t+1) = 0$ and $A_2(t+1) = 1$. $\Rightarrow \psi_2$ is the value of an "insurance policy" that pays 0 in state 1 and 1 in state 2.
- Consequently, ψ_1 and ψ_2 are called **state prices**.

- Notice from the first row of (1) that

$$\begin{aligned} 1 &= (1+r)\psi_1 + (1+r)\psi_2 \\ &= (1+r)(\psi_1+\psi_2) \\ \Rightarrow \psi_1+\psi_2 &= \frac{1}{1+r} \end{aligned} \tag{2}$$

$\Rightarrow$ Spending $\psi_1 + \psi_2$ today on the riskless asset would guarantee receiving 1 unit of account in period 2.

4. **Relevance of the theorem for asset pricing**

- We typically want to know how to price a derivative such as C, when we already know the values of the riskless asset B and the underlying asset A.
- Use the first two rows of equation (1) to determine (ψ_1, ψ_2):

$$\begin{bmatrix} 1 \\ A(t) \end{bmatrix} = \begin{bmatrix} (1+r) & (1+r) \\ A_1(t+1) & A_2(t+1) \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} (1+r) & (1+r) \\ A_1(t+1) & A_2(t+1) \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ A(t) \end{bmatrix}$$

- Use the third row of (1) to obtain $C(t)$:

$$C(t) = \psi_1 C_1(t+1) + \psi_2 C_2(t+1)$$

which is what we seek.

5. **Risk-neutral (risk-adjusted, synthetic) probabilities**

- Define

$$\begin{aligned} P_1^\star &= (1+r)\psi_1 \\ P_2^\star &= (1+r)\psi_2 \end{aligned}$$

- Rewrite equation (2) as

$$1 = p_1^\star + p_2^\star$$

- Positivity of state prices ψ_1 and $\psi_2 \Rightarrow P_i^\star > 0$.
- Taking these results together

$$\Rightarrow \begin{cases} 0 < P_i^\star \leq 1 \\ P_1^\star + P_2^\star = 1 \end{cases}$$

- We therefore can interpret the $P_i^\star$ as probabilities associated with the two states and can use them to calculate a type of expected value:
  - Write equation (1) as a system of three equations

$$\begin{aligned} 1 &= (1+r)\psi_1 + (1+r)\psi_2 && (3)\\ A(t) &= \psi_1 A_1(t+1) + \psi_2 A_2(t+1) && (4)\\ C(t) &= \psi_1 C_1(t+1) + \psi_2 C_2(t+1) && (5) \end{aligned}$$

  - Multiply RHS of (4) and (5) by $\frac{1+r}{1+r}$ to get

$$\begin{aligned} A(t) &= \frac{1}{1+r}\left[(1+r)\psi_1 A_1(t+1) + (1+r)\psi_2 A_2(t+1)\right]\\ &= \frac{1}{1+r}\left[P_1^\star A_1(t+1) + P_2^\star A_2(t+1)\right] && (6)\\ C(t) &= \frac{1}{1+r}\left[(1+r)\psi_1 C_1(t+1) + (1+r)\psi_2 C_2(t+1)\right]\\ &= \frac{1}{1+r}\left[P_1^\star C_1(t+1) + P_2^\star C_2(t+1)\right] && (7) \end{aligned}$$

- Inside the brackets we have expression that look like expected values. Indeed, they are called the expected values under the risk-neutral probabilities and are denoted $E^{\star}[A(t+1)]$ and $E^{\star}[C(t+1)]$.
- Outside the bracket we have $(1+r)^{-1}$, which is a one-period discount factor.
- The current asset prices $A(t)$ and $C(t)$ equal the present value of the expected values of $A(t+1)$ and $C(t+1)$ under the risk-neutral probabilities.
- Notice that the discounting is done with the risk-free rate even though the assets are risky.

6. **Comparison with true probabilities**

- The risk-neutral probabilities $P_i^\star$ generally do not equal the true probabilities P_i.
- The true expected values are

$$\begin{aligned} E[A(t+1)] &= P_1 A_1(t+1) + P_2 A_2(t+1) \\ E[C(t+1)] &= P_1 C_1(t+1) + P_2 C_2(t+1) \end{aligned}$$

- Usually (explained momentarily, and in more detail in the CAPM)

$$\begin{aligned} A(t) &< \frac{1}{1+r} E[A(t+1)] \\ C(t) &< \frac{1}{1+r} E[C(t+1)] \end{aligned}$$

in order to compensate for risk:

– Suppose otherwise, e.g.,

$$A(t) = \frac{1}{1+r} E[A(t+1)]$$

– Rearrange this as

$$1 + r = \frac{E[A(t+1)]}{A(t)}$$

$\Rightarrow$ return to A is riskless, which is a contradiction.

– The right relation between $1+r$ and $\frac{E[A(t+1)]}{A(t)}$ is

$$1 + r + \text{risk premium for } A = \frac{E[A(t+1)]}{A(t)}$$

– And similarly for $C(t)$:

$$1 + r + \text{risk premium for } C = \frac{E[C(t+1)]}{C(t)}$$

- Risk premiums are positive for assets that are positively correlated with "the market", which is most assets. If however, the correlation is negative, so will be the risk premium

$$
\begin{aligned}
\Rightarrow 1+r &> \frac{E[A(t+1)]}{A(t)} \\
\Leftrightarrow A(t) &> \frac{1}{1+r}E[A(t+1)]
\end{aligned}
$$

- These complications all disappear with risk-neutral expectations.

7. **Martingales and submartingales**

- **Definition**
  - **Martingale:** A random variable X_t is a martingale with respect to the probability P if it satisfies

$$E^P[X_{t+j}|I_t] = X_t \quad \forall j \geq 0$$

where I_t is the information set at time t.

  - **Submartingale:** X_t is a submartingale with respect to P if it satisfies

$$E^P[X_{t+j}|I_t] \geq X_t$$

- **Relevance**
  - Most asset prices, when discounted by the risk-free interest rate, will be submartingales under the true probability measure P but will be martingales under the risk-neutral measure $P^\star$:

$$\begin{aligned} A(t) &< E^P\left[\frac{A(t+j)}{(1+r)^j}|I_t\right] \\ A(t) &= E^{P^\star}\left[\frac{A(t+j)}{(1+r)^j}|I_t\right] \end{aligned}$$

    In terms of the general notation

$$\begin{aligned} X(t) &< E^P\left[X(t+j)|I_t\right] \\ X(t) &= E^{P^\star}\left[X(t+j)|I_t\right] \end{aligned}$$

  - There is a large body of mathematical theory concerning the properties of martingales that we can apply to discounted asset price if we use risk-neutral probabilities.

8. **Equal rates of return**

- Divide equations (6) and (7) by the current asset price and multiply by $(1+r)$ to obtain

$$
\begin{aligned}
(1+r) &= P_1^\star \frac{A_1(t+1)}{A(t)} + P_2^\star \frac{A_2(t+1)}{A(t)} \\
(1+r) &= P_1^\star \frac{C_1(t+1)}{C(t)} + P_2^\star \frac{C_2(t+1)}{C(t)}
\end{aligned}
$$

- The ratios $\frac{X_1(t+1)}{X_(t)}$ are the gross rates of return, so the RHS of each equation is the risk-neutral expected rate of return on the asset. They both equal $(1+r)$ and therefore are the same
- $\Rightarrow$ All risk-neutral expected rates of return equal the risk-free rate.

9. **No-arbitrage condition**

- Define the gross rates of return for states 1 and 2 as

$$\begin{aligned} R_1(t+1) &= \frac{A_1(t+1)}{A(t)} \\ R_2(t+1) &= \frac{A_2(t+1)}{A(t)} \end{aligned}$$

- Then we can write equation (3) and (4) as

$$\begin{aligned} 1 &= (1+r)\psi_1 + (1+r)\psi_2 \\ 1 &= R_1\psi_1 + R_2\psi_2 \end{aligned}$$

- Subtract the second equation from the first:

$$0 = \left[(1+r) - R_1\right]\psi_1 + \left[(1+r) - R_2\right]\psi_2 \tag{8}$$

- Recall that both ψ_1 and ψ_2 are positive. Then, we must have either

$$R_1 < 1 + r < R_2$$

or

$$R_2 < 1 + r < R_1$$

- Proof: Suppose $1 + r < R_1, R_2$, then both terms on RHS of (8) are negative which means RHS $<$ 0, which contradicts (8). If $R_1, R_2 < 1 + r$, then RHS $>$ 0.
- Intuition is straightforward.
  - If $1 + r < R_1, R_2$, then there is an arbitrage opportunity. You can borrow at rate r, invest in the asset and guarantee a positive return $\geq \min(R_1, R_2)$.
  - If $R_1, R_2 < 1 + r$, then you can short the asset and lend at rate r, guaranteeing a profit.

10. **Dividends**

- Suppose the asset A pays a dividend in period $t+1$ equal to a fraction $d(t)$ of its price in $t+1$:

$$div(t+1) = d(t) \times A(t+1)$$

where the dividend rate is determined in period t and therefore is known.

- the system of equation becomes

$$\begin{bmatrix} 1 \\ A(t) \\ C(t) \end{bmatrix} = \begin{bmatrix} (1+r) & (1+r) \\ [1+d(t)]\,A_1(t+1) & [1+d(t)]\,A_2(t+1) \\ C_1(t+1) & C_2(t+1) \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$$

- The first and third equations are the same as before.
- The second equation differs slightly from before:

$$A(t) = [1 + d(t)]\,[\psi_1 A_1(t+1) + \psi_2 A_2(t+1)]$$

- We can proceed as before: multiply RHS by $\frac{1+r}{1+r}$ to get

$$A(t) = \frac{1 + d(t)}{1 + r}\,[P_1^\star A_1(t+1) + P_2^\star A_2(t+1)]$$