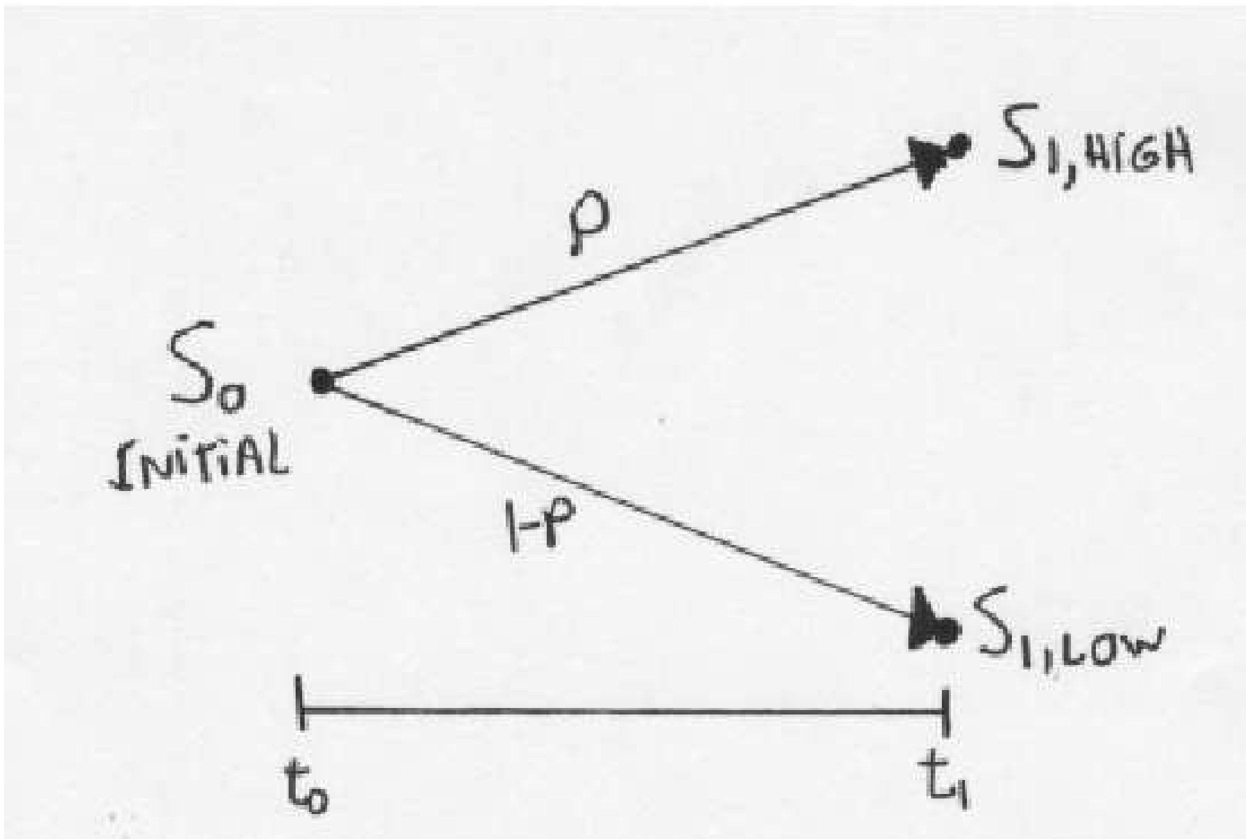
10 Binomial Trees

10.1 One-step model

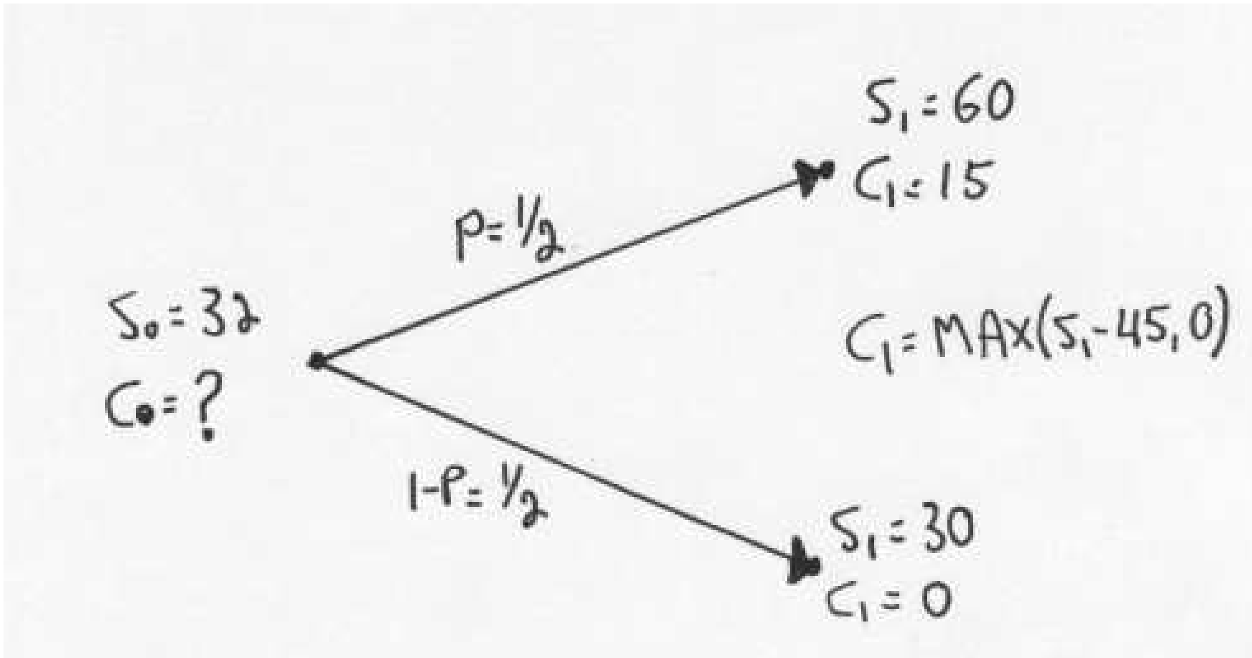
1. **Model structure**



- There is only one time interval (t_0, t_1)
- There are only two possible outcomes for S
  - High, with probability p
  - Low, with probability $1 - p$
- In the diagram above, Low is shown below Initial and High is above Initial, implying that one of the outcome is an increase in value and the other is a decrease.
- In general however, both outcomes could be increases or decreases. We study the general model later.

2. Pricing a call option: numerical example

(a) Setup



- The risk-free rate is $r = 0.25$
- The call option's strike price is $K = 45$.

(b) **Notice that S_0 does not equal the discounted expected value of S_1:**

$$
\begin{aligned}
S_0 &= 32 \\
&< \frac{1}{1+r}E[S_1] \\
&= \frac{1}{1+0.25}\left[p \times 60 + (1-p) \times 30\right] \\
&= \frac{4}{5}\left[\frac{1}{2}60 + \frac{1}{2}30\right] \\
&= 36
\end{aligned}
$$

Presumably, this fact reflects risk aversion.

(c) Pricing the call: riskless portfolio method

- Consider the following portfolio:
  - long Δ shares of stock
  - Short one call
- Find the value of Δ that makes the portfolio riskless
  - If the stock rises to 60, the portfolio is worth $60\Delta - 15$
  - If the stock fall to 30, the portfolio is worth 30Δ.
  - To be riskless, the portfolio must give the same value for either outcome:

$$60\Delta - 15 = 30\Delta \;\Rightarrow\; \Delta = 1/2$$

  - Value of this riskless portfolio at t_1: 15

- In the absence of arbitrage, riskless portfolio must earn the risk-free rate of interest.
- The value of this portfolio at t_0 is then

$$\begin{aligned} \frac{1}{1+r}15 &= \frac{4}{5}15 \\ &= 12 \\ &= S_0\Delta - C_0 \end{aligned}$$

- But $S_0 = 32$ and $\Delta = 1/2$. This means that

$$12 = 32 \times \frac{1}{2} - C_0 \;\Rightarrow\; C_0 = 4$$

(d) Pricing the call: risk-neutral valuation method

- We know that

$$\begin{aligned} C_0 &= \frac{1}{1+r} E^\star[C_1] \\ &= \frac{1}{1+r} \left[p^\star C_{1,high} + (1-p^\star) C_{1,low}\right] \\ &= \frac{4}{5} \left[p^\star \times 15 + (1-p^\star) \times 0\right] \\ &= 12p^\star \end{aligned}$$

- All we need is $p^\star$

- We get $p^\star$ form the expression for the value of the stock:

$$
\begin{aligned}
S_0 &= \frac{1}{1+r} E^\star[S_1] \\
32 &= \frac{4}{5}\left[p^\star \times 60 + (1-p^\star)30\right] \\
&= \frac{4}{5}\left[30 \times p^\star + 30\right] \\
p^\star &= \frac{1}{3}
\end{aligned}
$$

Note that $p^\star \neq p = 1/2$.

- Therefore, $C_0 = 12 \times 1/3 = 4$.
- The advantage to the risk-neutral approach is that we can use it to price **any** derivative once we have computed $p^\star$.

3. **Pricing a put option**

- Suppose we have a put option on the same stock with the same strike price $K = 45$.
- We have

$$\begin{aligned} P_1 &= \max[K - S_1, 0] \\ &= \max[45 - S_1] \end{aligned}$$

- Therefore,

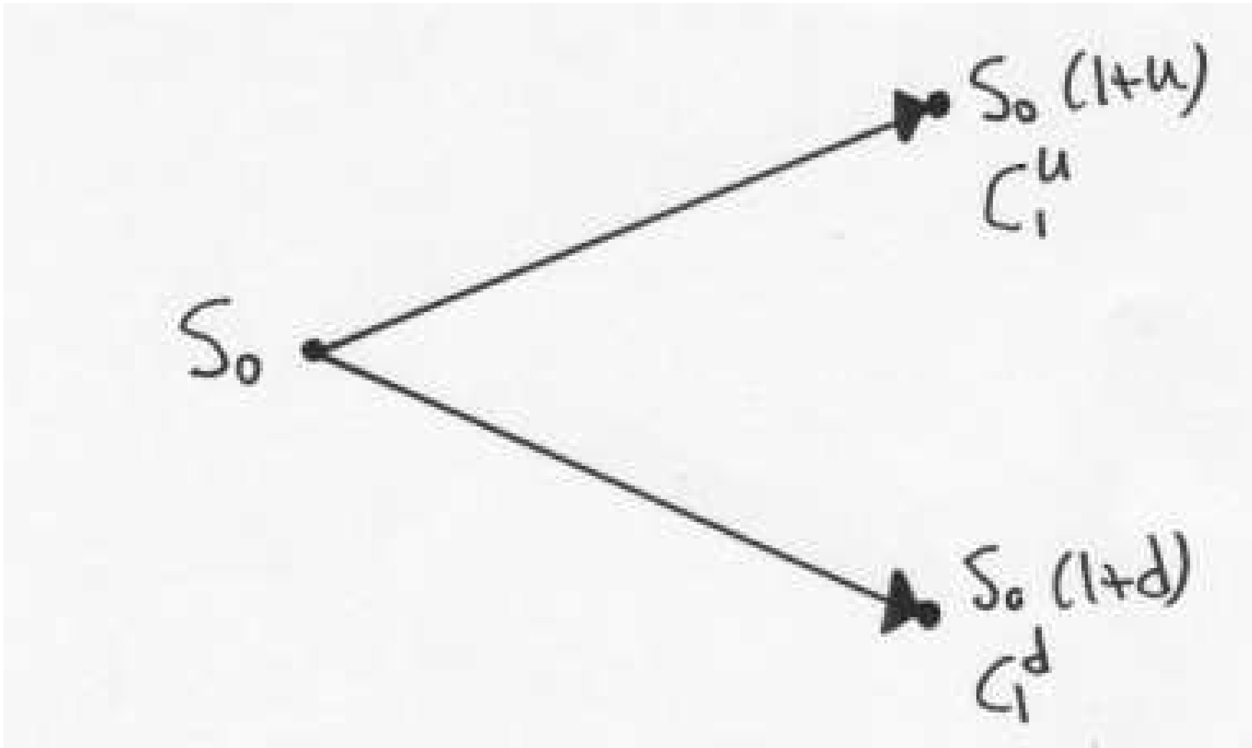
$$\begin{aligned} P_{1,high} &= \max[45 - 60, 0] \\ &= 0 \\ P_{1,low} &= \max[45 - 30, 0] \\ &= 15 \end{aligned}$$

- The value of the put at t_0 is then

$$
\begin{aligned}
P_0 &= \frac{4}{5}\left[\frac{1}{3} \times 0 + \frac{2}{3} \times 15\right] \\
&= \frac{4}{5}[10] \\
&= 8
\end{aligned}
$$

4. **Relation between the two pricing methods**

   - Consider the binomial tree model for the call option



where $d < u$ but otherwise are unrestricted

- Riskless portfolio approach
  - If S_0 rises to $S_0(1+u)$, the portfolio at time t is worth $S_0(1+u)\Delta - C_1^u$
  - If S_0 falls to $S_0(1+d)$, the value is $S_0(1+d)\Delta - C_1^d$.
  - Equating these two gives

$$
\begin{aligned}
S_0(1+d)\Delta - C_1^u &= S_0(1+d)\Delta - C_1^d \\
\Rightarrow \Delta &= \frac{C_1^u - C_1^d}{S_0(u-d)}
\end{aligned}
$$

- The value of the portfolio in t_1 is $S_0(1+u)\Delta - C_1^u$ (or $S_0(1+d)\Delta - C_1^d$)
- Since the portfolio is riskless, its value at t_0 is

$$\frac{1}{1+r}\left[S_0(1+u)\Delta - C_1^u\right]$$

- Which must equal our other expression for the t_0-value: $S_0\Delta - C_0$. That is,

$$\begin{aligned} S_0\Delta - C_0 &= \frac{1}{1+r}\left[S_0(1+u)\Delta - C_1^u\right] \\ C_0 &= S_0\Delta - \frac{1}{1+r}\left[S_0(1+u)\Delta - C_1^u\right] \end{aligned}$$

- Substitute for Δ

$$\begin{aligned} C_0 &= S_0\left(\frac{C_1^u - C_1^d}{S_0(u-d)}\right) - \frac{1}{1+r}\left[S_0(1+u)\left(\frac{C_1^u - C_1^d}{S_0(u-d)}\right) - C_1^u\right] \\ &= \frac{1}{1+r}\left[\frac{r-d}{u-d}C_1^u - \frac{r-u}{u-d}C_1^d\right] \end{aligned}$$

- Obtaining the risk-neutral probabilities

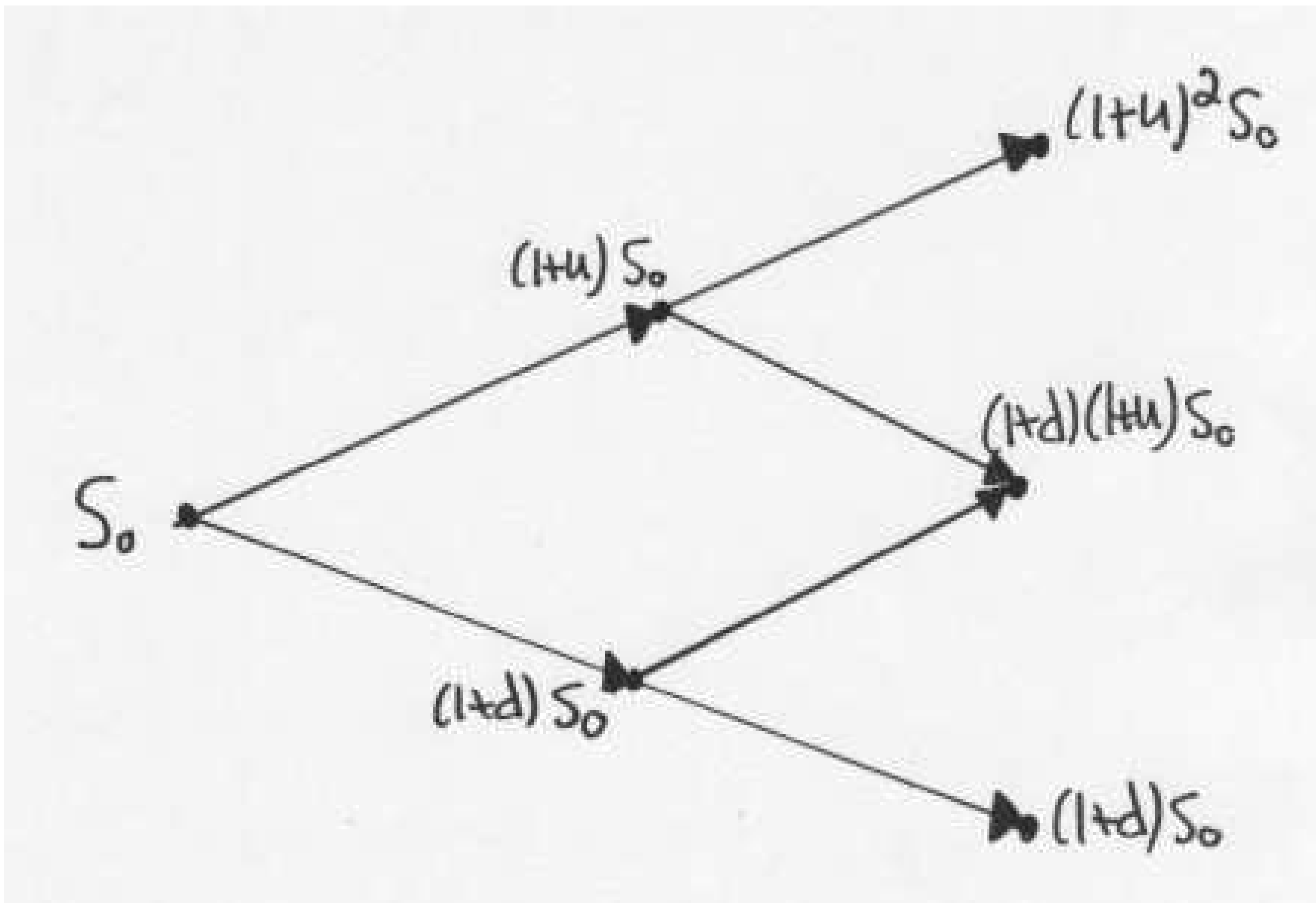
$$
\begin{aligned}
C_0 &= \frac{1}{1+r}\left[\frac{r-d}{u-d}C_1^u - \frac{r-u+d-d}{u-d}C_1^d\right] \\
&= \frac{1}{1+r}\left[\underbrace{\frac{r-d}{u-d}}_{\equiv p^\star}C_1^u + \left(1-\frac{r-d}{u-d}\right)C_1^d\right]
\end{aligned}
$$

5. **Irrelevance of true probabilities and expectations**

   - The asset pricing formulas do not use p or anything dependent on it, such as $E[S_1]$.
   - In the real world, p interacts with the risk averse characteristics of the utility function to determine values.
   - The asset pricing formulas avoid the complication of utility function curvature (which is probably different for each person) by transforming the problem to one of risk neutrality of a representative agent.

10.2 Multi-step models

1. **Two-step model**



- Start with period 2 and use the preceding methods to determine S_1. For example,

$$(1+u)S_0 = \frac{1}{1+r}\left[p^\star(1+u)^2 S_0 + (1-p^\star)(1+d)(1+u)S_0\right]$$

- Calculate $p^\star$ (which way change at each step)
- Then use these values for period 1 to work back to S_0.
- Use $p^\star$ to calculate the price of any derivative.

2. **Multi-step model**

   Obvious generalization

10.3 American options

Follow the same procedure, except that at each node we compare the value obtained from the preceding method with the payment at that node from early exercise. The greater of those two number is the value at that node.

10.4 Delta

- The delta of a stock option is the ratio of the difference between the two option values at the end of the period to the difference between the two stock values at the same time:

$$\begin{aligned} \Delta &= \frac{C_1^u - C_1^d}{S_0 u - S_0 d} \\ &= \frac{C_1^u - C_1^d}{S_0(u-d)} \end{aligned}$$

This is the same Δ we used earlier to create a riskless portfolio.

- Picking Δ to create a riskless portfolio is called **delta hedging**.
- In multi-period models, delta generally changes over time $\Rightarrow$ continuous portfolio rebalancing.