15 CAPM and portfolio management

15.1 Theoretical foundation for mean-variance analysis

- We assume that investors try to maximize the expected utility, $E[U(W)]$, where W is their wealth.
- Consider a Taylor expansion of the utility function around $E[W]$ (the expected wealth):

$$\begin{aligned} U(W) &= U(E[W]) + U'(E[W])\,(W - E[W]) \\ &\quad + \frac{1}{2}U''(E[W])\,(W - E[W])^2 \\ &\quad + \sum_{n=3}^{\infty} \frac{1}{n!}U^{(n)}(E[W])\,(W - E[W])^n \end{aligned}$$

- Next, take the expectation:

$$\begin{aligned} E[U(W)] &= U(E[W]) + U'(E[W])E\left[(W - E[W])\right] \\ &+\frac{1}{2}U''(E[W])E\left[(W - E[W])^2\right] \\ &+\sum_{n=3}^{i} nfty\frac{1}{n!}U^{(n)}(E[W])E\left[(W - E[W])^n\right] \end{aligned}$$

- With a mean-variance analysis we stop at the second order.
- There are two cases where this can be justified:
  - If W is normally distributed, then the first two moments characterize all the moments.
  - If the utility is quadratic, then $U(W)^{(n)} = 0$ for $n \geq 3$.

15.2 Portfolio with minimum variance

15.2.1 Simple case: 2 assets

- Consider the following two assets x_1 and x_2 with
  - $Var[x_1] = \sigma_1^2, Var[x_2] = \sigma_2^2$ and $Cov[x_1, x_2] = \sigma_{12}$
  - w_1: the weight of the first asset in the portfolio
  - $1 - w_1$: the weight of the second asset in the portfolio
- Denote by σ_p^2 the variance of the portfolio:

$$\begin{aligned} \sigma_p^2 &= Var[w_1 x_1 + (1 - w_1) x_2] \\ &= w_1^2 \sigma_1^2 + (1 - w_1)^2 \sigma_2^2 + 2 w_1 (1 - w_1) \sigma_{12} \end{aligned}$$

- We want to find the minimum variance portfolio: $\min_{w_1} \sigma_p^2$
- The First Order Condition (FOC) is:

$$2w_1^\star\sigma_1^2 - 2(1-w_1^\star)\sigma_2^2 + 2(1-w_1^\star)\sigma_{12} + 2w_1^\star(-1)\sigma_{12} \quad = \quad 0$$

- Solving for $w_1^\star$, we get

$$w_1^\star = \frac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}$$

- Diversification principle:
  - Look at the FOC when we take $w_1 = 0$:

$$\begin{aligned}\frac{\partial \sigma_p^2}{\partial w_1}(w_1 = 0) &= 2(\sigma_{12} - \sigma_2^2) \\ &= 2(\rho_{12}\sigma_1\sigma_2 - \sigma_2^2) \\ &= 2\sigma_1\sigma_2\left(\rho_{12} - \frac{\sigma_2}{\sigma_1}\right)\end{aligned}$$

  - If $\rho_{12} < 0$ or if $\rho_{12} > 0$ but $\sigma_2/\sigma_1 > \rho_{12}$, then $\frac{\partial \sigma_p^2}{\partial w_1}(w_1 = 0) < 0$ so I should increase w_1 (i.e. buying x_1).
  - If $\rho_{12} > 0$ but $\sigma_2/\sigma_1 < \rho_{12}$, then $\frac{\partial \sigma_p^2}{\partial w_1}(w_1 = 0) > 0$ so I should decrease w_1 (i.e. short-sell x_1).

15.2.2 Case with N assets

- Consider the following elements:
  - $w = (w_1, w_2, \ldots, w_N)'$: portfolio weights
  - $x = (x_1, x_2, \ldots, x_N)'$: asset returns
  - $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_N)'$: expected asset returns
  - the variance matrix

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1N} \\ \sigma_{12} & \sigma_2^2 & \cdots & \sigma_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1N} & \sigma_{2N} & \cdots & \sigma_N^2 \end{bmatrix}$$

  - $x_p = w'x$: portfolio's return
  - $\sigma_p^2 = w'\Sigma w$: portfolio's variance

- Define the following elements:
  - i: $N \times 1$ vector of 1.
  - $A = i'\Sigma^{-1}\bar{x}$
  - $B = \bar{x}'\Sigma^{-1}\bar{x}$
  - $C = i'\Sigma^{-1}i$
  - $D = BC - A^2$ (we can show that $D > 0$)
- Show that $w'\Sigma w$ is the variance of the portfolio.

- We want to characterize the mean-variance frontier (finding the portfolio with the lowest variance for a given expected return)
- The problem is

$$\begin{aligned} & \min_w \frac{1}{2} w' \Sigma w \\ \text{subject to} \quad & i'w = 1 \\ & \bar{x}'w = \mu \end{aligned}$$

- This is a constrained optimization
- We write the Lagrangian

$$L = \frac{1}{2} w' \Sigma w + \gamma (1 - i'w) + \lambda (\mu - \bar{x}'w)$$

- The FOCs are

$$\frac{\partial L}{\partial w} = \begin{bmatrix} \partial L / \partial w_1 \\ \partial L / \partial w_2 \\ \vdots \\ \partial L / \partial w_N \end{bmatrix} = \Sigma w - \gamma i - \lambda \bar{x} = 0 \qquad (1)$$

$$\frac{\partial L}{\partial \gamma} = 1 - i'w = 0 \qquad (2)$$

$$\frac{\partial L}{\partial \lambda} = \mu - \bar{x}'w = 0 \qquad (3)$$

- From equation (1), we get

$$w^\star = \gamma\Sigma^{-1}i + \lambda\Sigma^{-1}\bar{x}$$

- We need to solve for γ and λ
- From equation (2), we know that $1 - i'w^\star = 0$

$$\begin{aligned} 1 - i'[\gamma\Sigma^{-1}i + \lambda\Sigma^{-1}\bar{x}] &= 0 \\ 1 - \gamma i'\Sigma^{-1}i - \lambda i'\Sigma^{-1}\bar{x} &= 0 \\ 1 - \gamma C - \lambda A &= 0 \qquad (4) \end{aligned}$$

- From equation (3), we know that $\mu - \bar{x}'w^\star = 0$

$$\begin{aligned} \mu - \bar{x}'[\gamma\Sigma^{-1}i + \lambda\Sigma^{-1}\bar{x}] &= 0 \\ \mu - \gamma\bar{x}'\Sigma^{-1}i - \lambda\bar{x}'\Sigma^{-1}\bar{x} &= 0 \\ \mu - \gamma A - \lambda B &= 0 \qquad (5) \end{aligned}$$

- We can solve (4) and (5) for γ and λ. We get

$$\begin{aligned} \lambda &= \frac{C\mu - A}{D} \\ \gamma &= \frac{B - A\mu}{D} \end{aligned}$$

- It follows that the optimal portfolio is

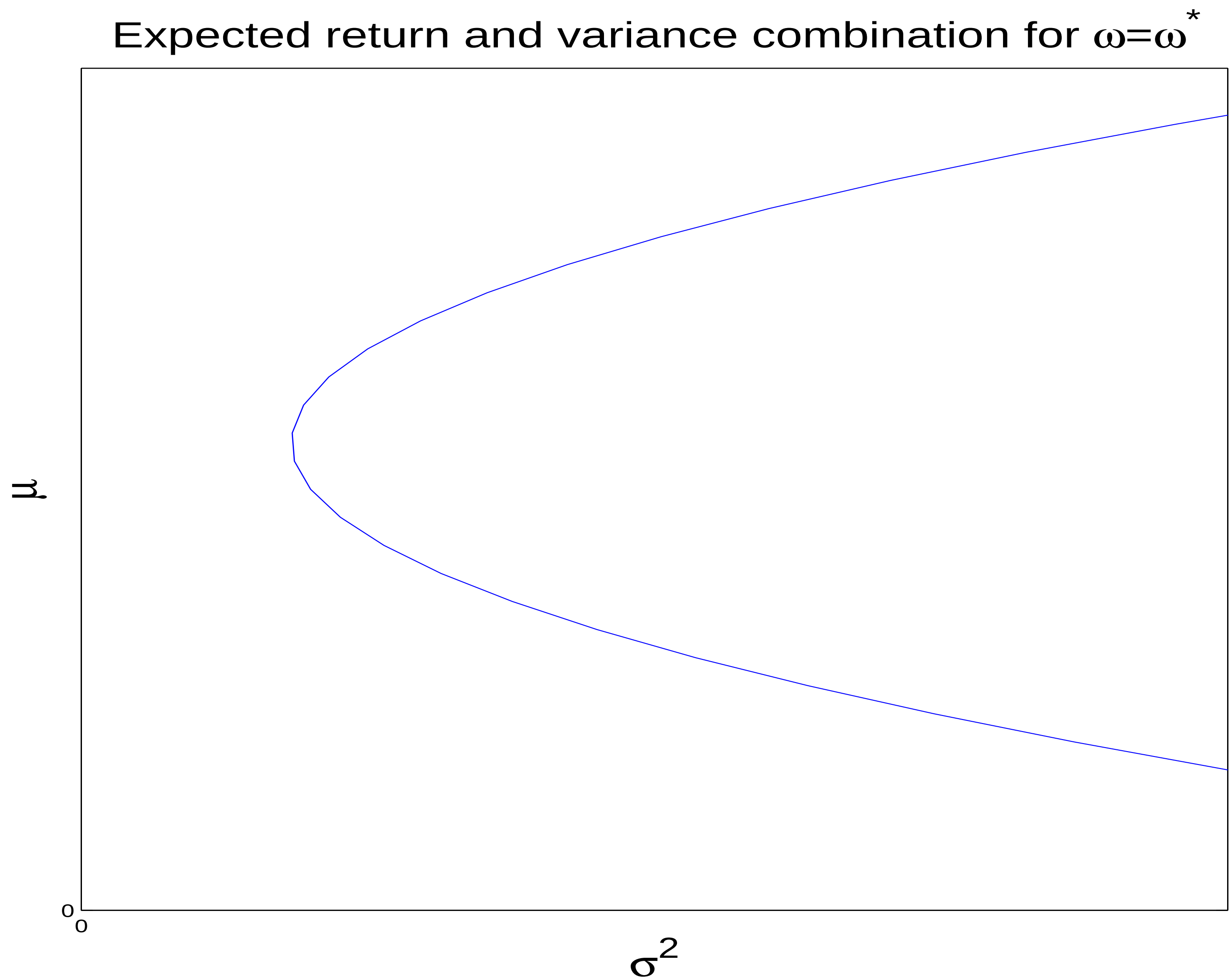
$$\begin{aligned} w^\star &= \left[\frac{B - A\mu}{D}\right]\Sigma^{-1}i + \left[\frac{C\mu - A}{D}\right]\Sigma^{-1}\bar{x} \\ &= \left[\frac{B\Sigma^{-1}i}{D} - \frac{A\Sigma^{-1}\bar{x}}{D}\right] + \left[\frac{C\Sigma^{-1}\bar{x}}{D} - \frac{A\Sigma^{-1}i}{D}\right]\mu \\ &= g + h\mu \end{aligned}$$

where

$$\begin{aligned} g &= \frac{1}{D}\left[B\Sigma^{-1}i - A\Sigma^{-1}\bar{x}\right] \\ h &= \frac{1}{D}\left[C\Sigma^{-1}\bar{x} - A\Sigma^{-1}i\right] \end{aligned}$$

- Variance of the portfolio when $w = w^\star$

$$
\begin{aligned}
\sigma_p^2 &= w^{\star\prime}\Sigma w^\star \\
&= w^{\star\prime}\Sigma(\gamma\Sigma^{-1}i + \lambda\Sigma^{-1}\bar{x}) \\
&= w^{\star\prime}\left[\gamma\Sigma\Sigma^{-1}i + \lambda\Sigma\Sigma^{-1}\bar{x}\right] \\
&= w^{\star\prime}\left[\gamma i + \lambda\bar{x}\right] \\
&= \gamma \underbrace{w^{\star\prime}i}_{=1} + \lambda \underbrace{w^{\star\prime}\bar{x}}_{=\mu} \\
&= \gamma + \lambda\mu \\
&= \frac{B - A\mu}{D} + \left(\frac{C\mu - A}{D}\right)\mu \\
&= \frac{B - 2A\mu + C\mu^2}{D} \quad \Rightarrow \text{ parabola}
\end{aligned}
$$

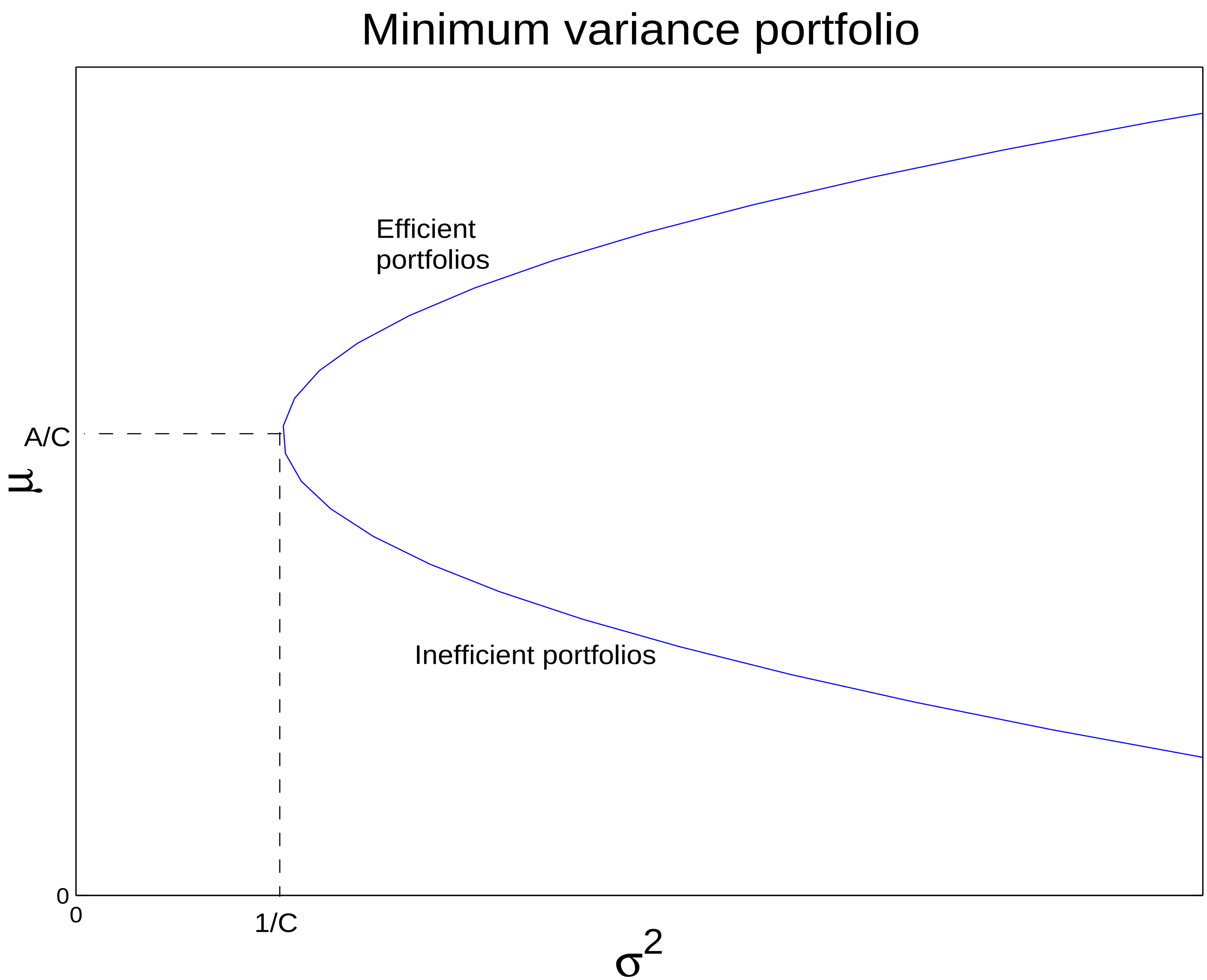

Expected return and variance combination for ω=ω*
μ
0
0
σ2

- We can find the global minimal variance

$$\frac{\partial \sigma_p^2}{\partial \mu} = \frac{2C\mu - 2A}{D} = 0 \quad \Rightarrow \quad \mu_g = \frac{A}{C}$$

- What is this variance?

$$\begin{aligned}(\sigma_p^2)_g &= \frac{B - 2A\frac{A}{C} + C\left(\frac{A}{C}\right)^2}{D} \\ &= \frac{BC - 2A^2 + A^2}{CD} \\ &= \frac{BC - A^2}{CD} \\ &= \frac{1}{C}\end{aligned}$$


Minimum variance portfolio
Efficient
portfolios
Inefficient portfolios
A/C
μ
0
0
1/C
σ2

- What is γ and λ for this μ_g?

$$\begin{aligned} \lambda_g &= \frac{C\left(\frac{A}{C}\right) - A}{D} = 0 \quad \text{expected return constraint not binding} \\ \gamma_g &= \frac{B - A\left(\frac{A}{C}\right)}{D} = \frac{1}{C} \end{aligned}$$

- What is the portfolio with minimum global variance?

$$\begin{aligned} w_g &= \gamma_g \Sigma^{-1} i + \lambda_g \Sigma^{-1} \bar{x} \\ &= \frac{1}{C} \Sigma^{-1} i + 0 \Sigma^{-1} \bar{x} \\ &= \frac{1}{C} \Sigma^{-1} i \\ &= \frac{\Sigma^{-1} i}{i' \Sigma^{-1} i} \end{aligned}$$

- If we go back to $w^\star$ (optimal portfolio for a given expected return):

$$\begin{aligned} w^\star &= \gamma\Sigma^{-1}i + \lambda\Sigma^{-1}\bar{x} \\ &= \gamma C \underbrace{\left(\frac{\Sigma^{-1}i}{C}\right)}_{=w_g} + \lambda A \underbrace{\left(\frac{\Sigma^{-1}\bar{x}}{A}\right)}_{=w_d} \end{aligned}$$

  We see that $w^\star$ is a combination of:

  - The portfolio with the lowest global variance but lowest expected return (w_g).
  - A second portfolio (w_d) that will increase expected return but will increase the variance.

- But what is $\gamma C + \lambda A$?

$$\begin{aligned}
\gamma C + \lambda A &= \left(\frac{B - A\mu}{D}\right) C + \left(\frac{C\mu - A}{D}\right) A \\
&= \frac{BC - AC\mu}{D} + \frac{AC\mu - A^2}{D} \\
&= \frac{BC - A^2}{D} \\
&= 1
\end{aligned}$$

Conclusion: (γC) and (λA) are the two shares of a portfolio

15.3 Covariance properties of minimal variance portfolios

- w_g has a covariance constant with every asset or portfolio ($= 1/C$):

$$
\begin{aligned}
Cov(x_g, x_p) &= E\left[w_g'(x-\bar{x})(x-\bar{x})'w_p\right] \\
&= w_g' E[(x-\bar{x})(x-\bar{x})'] w_p \\
&= w_g' \Sigma w_p \\
&= \left(\frac{i'\Sigma^{-1}}{C}\right) \Sigma w_p \\
&= \frac{i' w_p}{C} \\
&= \frac{1}{C}
\end{aligned}
$$

- Covariance of portfolio w_d with any other portfolio:

$$
\begin{aligned}
Cov(x_d, x_p) &= E\left[w_d'(x-\bar{x})(x-\bar{x})'w_p\right] \\
&= w_d'\Sigma w_p \\
&= \frac{\bar{x}\Sigma^{-1}}{A}\Sigma w_p \\
&= \frac{\bar{x}w_p}{A} \\
&= \frac{\bar{x}_p}{A}
\end{aligned}
$$

We see that the expected return of any portfolio will be proportional to its covariance with w_d since $\bar{x}_p = A\,Cov(x_d, x_p)$.

- Consider a portfolio a on the minimum variance frontier ($w_a = (1-a)w_g + aw_d$). What is the covariance between a and another portfolio p?

$$\begin{aligned} Cov(x_a, x_p) &= (1-a)Cov(x_g, x_p) + aCov(x_d, x_p) \\ &= (1-a)\frac{1}{C} + a\frac{\bar{x}_p}{A} \end{aligned}$$

If $x_p = x_a$, then

$$Cov(x_a, x_p) = Var(x_a) = \frac{1-a}{C} + \frac{a}{A}\bar{x}_a$$

As long as a is not the minimum variance portfolio, it's possible to find a portfolio z that has a zero covariance with a:

$$\begin{aligned} Cov(x_a, x_z) &= \frac{1-a}{C} + a\frac{\bar{x}_z}{A} = 0 \\ &\Rightarrow \bar{x}_z = \frac{a-1}{a}\frac{A}{C} \end{aligned}$$

This is the expected return of a portfolio with zero covariance with any portfolio on the minimum variance frontier.

- We saw previously that

$$\begin{aligned} Var(x_a) &= \frac{1-a}{C} + \frac{a}{A}\bar{x}_a \\ &= -\frac{a\bar{x}_z}{A} + \frac{a}{A}\bar{x}_a \quad \text{using the result from previous slide} \\ &= \frac{a}{A}(\bar{x}_a - \bar{x}_z) \end{aligned}$$

- Next, define

$$\beta_{pa} \equiv \frac{Cov(x_a, x_p)}{Var(x_a)}$$

- Using previous results

$$\begin{aligned}
\beta_{pa} &= \frac{\frac{1-a}{C} + \frac{a}{A}\bar{x}_p}{\frac{a}{A}(\bar{x}_a - \bar{x}_z)} \\
\beta_{pa}(\bar{x}_a - \bar{x}_z) &= \frac{A}{a}\left[\frac{1-a}{C} + \frac{a}{A}\bar{x}_p\right] \\
&= \frac{A}{a}\left[\frac{(1-a)A + aC\bar{x}_p}{CA}\right] \\
&= \underbrace{\frac{1-a}{a}\frac{A}{C}}_{=-\bar{x}_z} + \bar{x}_p \\
\bar{x}_p &= \bar{x}_z + \beta_{pa}(\bar{x}_a - \bar{x}_z)
\end{aligned}$$

- The last equality is the CAPM equation without a risk-free asset.
- It is telling us that the expected return on any portfolio p (i.e. $\bar{x}_p$) is equal to the expected return on a portfolio uncorrelated with portfolio a (i.e. $\bar{x}_z$) plus β_{pa} times the excess return of a over z.

Portfolio a is a portfolio on the minimum variance frontier. Portfolio z is a portfolio uncorrelated with portfolio a.

15.4 Introduction of a riskless asset

- Now assume there is one more asset. This asset is riskless and has a risk-free rate r_f.
- The problem is now

$$\min_w \frac{1}{2} w' \Sigma w$$

subject to

$$w' \bar{x} + (1 - w' i) r_f = \mu$$

- We form the Lagrangian

$$L = \frac{1}{2}w'\Sigma w + \lambda(\mu - w'\bar{x} - (1 - w'i)r_f)$$

- The FOCs are:

$$\frac{\partial L}{\partial w} = \Sigma w + \lambda(-\bar{x} + ir_f) = 0 \quad (6)$$

$$\frac{\partial L}{\partial \lambda} = \mu - w'\bar{x} - (1 - w'i)r_f = 0 \quad (7)$$

- In equation (6) we can solve for w:

$$w^\star = \lambda\Sigma^{-1}(\bar{x} - ir_f) \quad (8)$$

- We can solve for λ using equation (7):

$$
\begin{aligned}
\mu &= w^{\star\prime}\bar{x} + (1 - w^{\star\prime} i) r_f \\
\mu &= [\lambda\Sigma^{-1}(\bar{x} - i r_f)]'\bar{x} + (1 - [\lambda\Sigma^{-1}(\bar{x} - i r_f)]' i) r_f \\
\mu &= \lambda(\bar{x}' - r_f i')\Sigma^{-1}\bar{x} + (1 - \lambda(\bar{x}' - r_f i'\Sigma^{-1} i)) r_f \\
\mu &= \lambda(\bar{x}'\Sigma^{-1}\bar{x} - r_f i'\Sigma^{-1}\bar{x}) + r_f - \lambda(\bar{x}'\Sigma^{-1} i - r_f i'\Sigma^{-1} i) r_f \\
\mu &= \lambda(B - r_f A) + r_f - \lambda(A - r_f C) r_f \\
\mu - r_f &= \lambda(B - 2A r_f + C r_f^2) \\
\lambda &= \frac{\mu - r_f}{H} \qquad (9)
\end{aligned}
$$

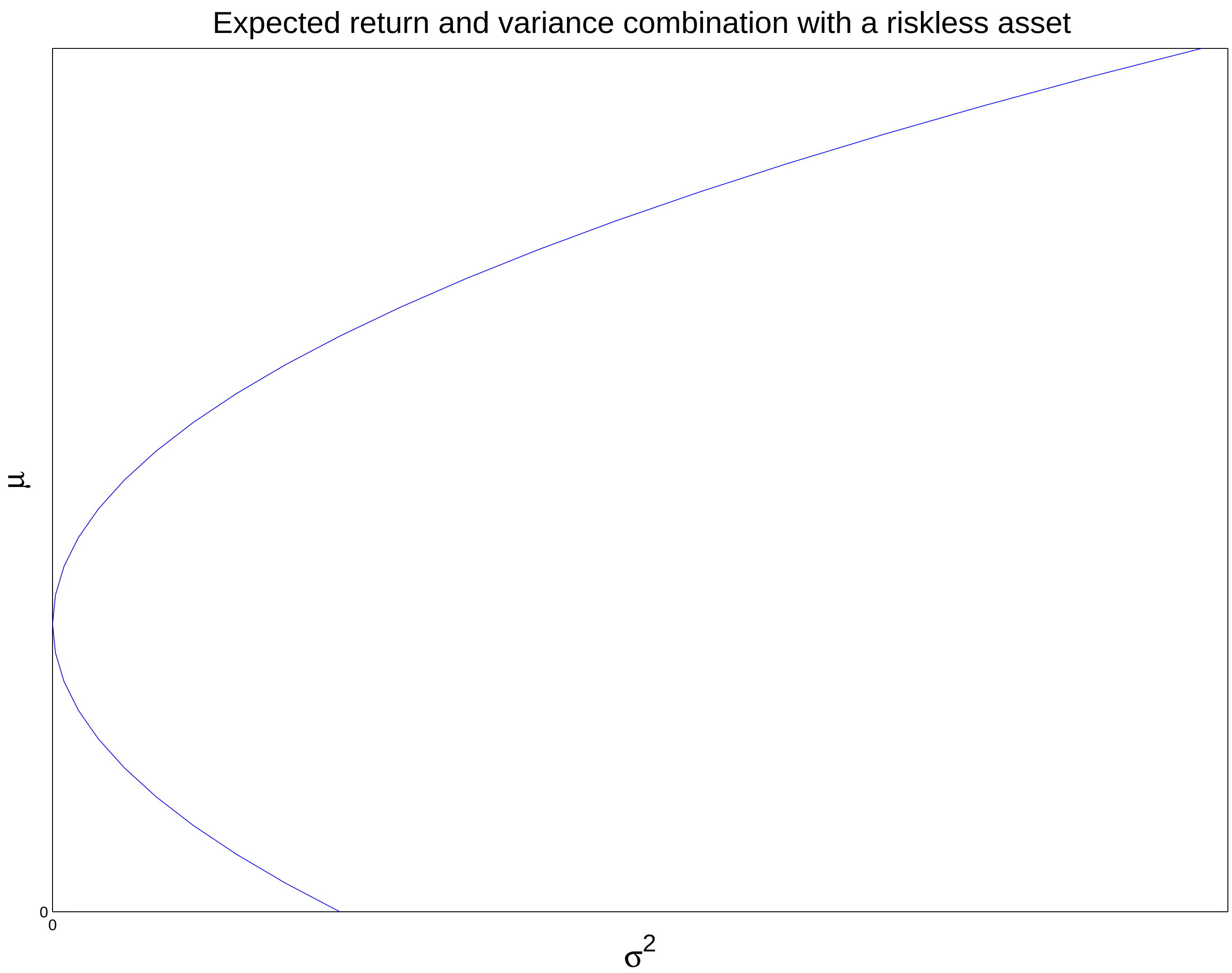
where $H = B - 2Ar_f + Cr_f^2$

- Equation (9) into equation (8):

$$
w^\star = \Sigma^{-1}(\bar{x} - i r_f)\left(\frac{\mu - r_f}{H}\right)
$$

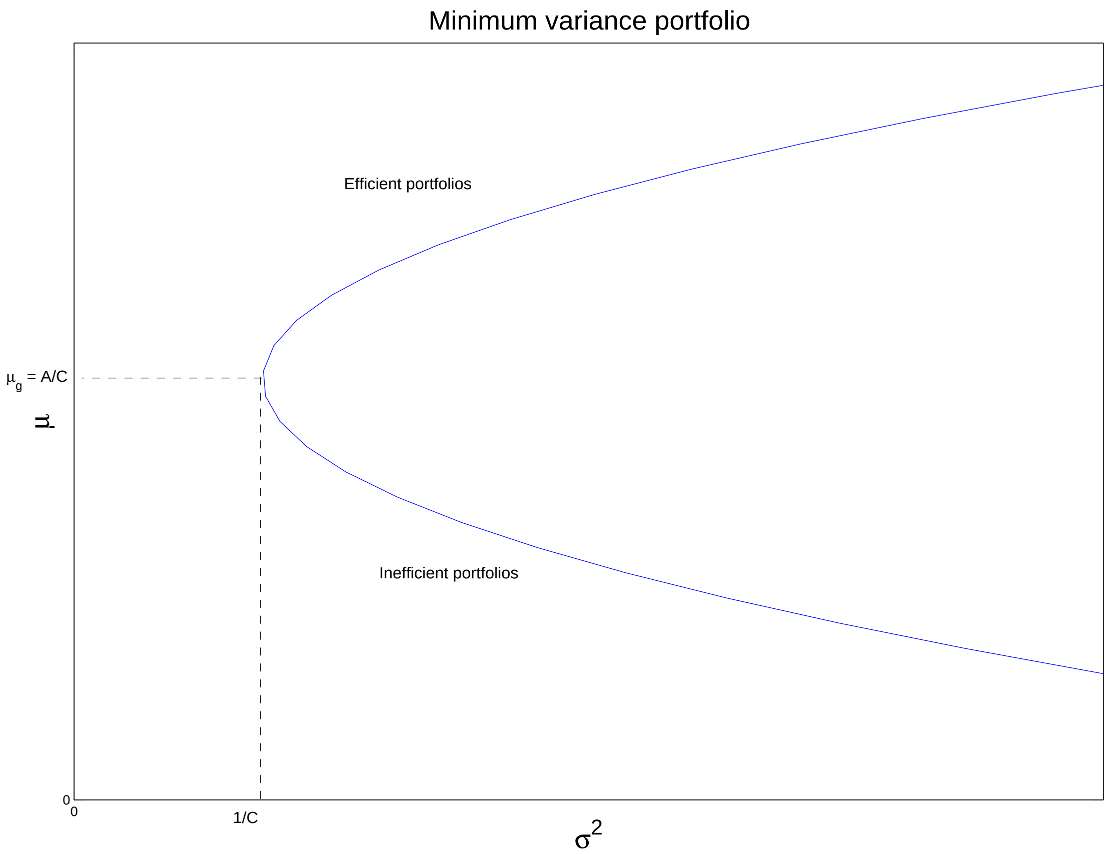
- The variance of this portfolio is

$$
\begin{aligned}
\sigma_p^2 &= w^{\star\prime}\Sigma w^{\star} \\
&= w^{\star\prime}\Sigma\left[\Sigma^{-1}(\bar{x}-ir_f)\left(\frac{\mu-r_f}{H}\right)\right] \\
&= w^{\star\prime}\left[(\bar{x}-ir_f)\left(\frac{\mu-r_f}{H}\right)\right] \\
&= (\underbrace{w^{\star\prime}\bar{x}}_{=\mu}-\underbrace{w^{\star\prime}i}_{=1}r_f)\frac{(\mu-r_f)}{H} \\
&= \frac{(\mu-r_f)^2}{H}
\end{aligned}
$$

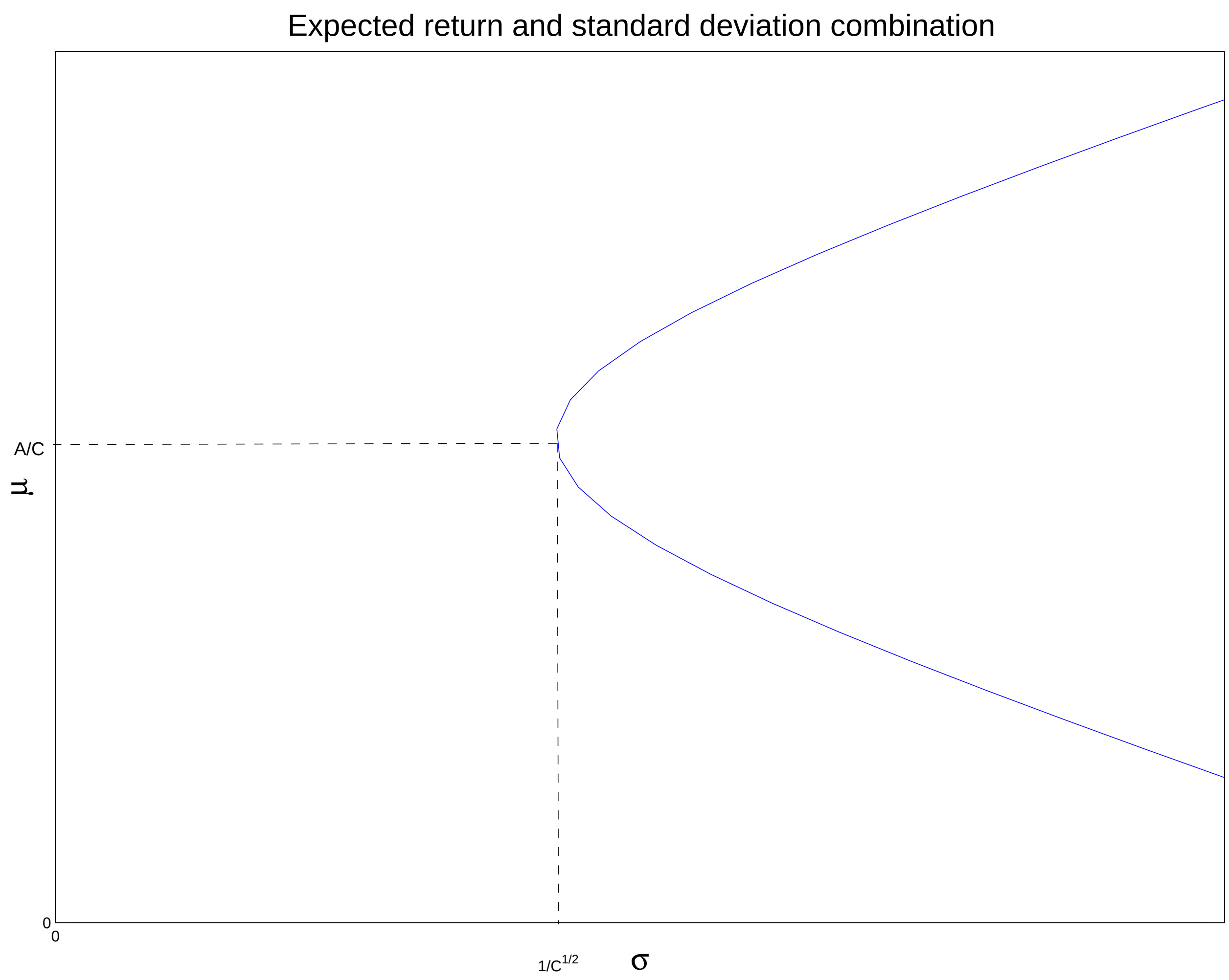

Expected return and variance combination with a riskless asset
μ
0
0
σ2

- Recall the parabola when we have N risky assets:

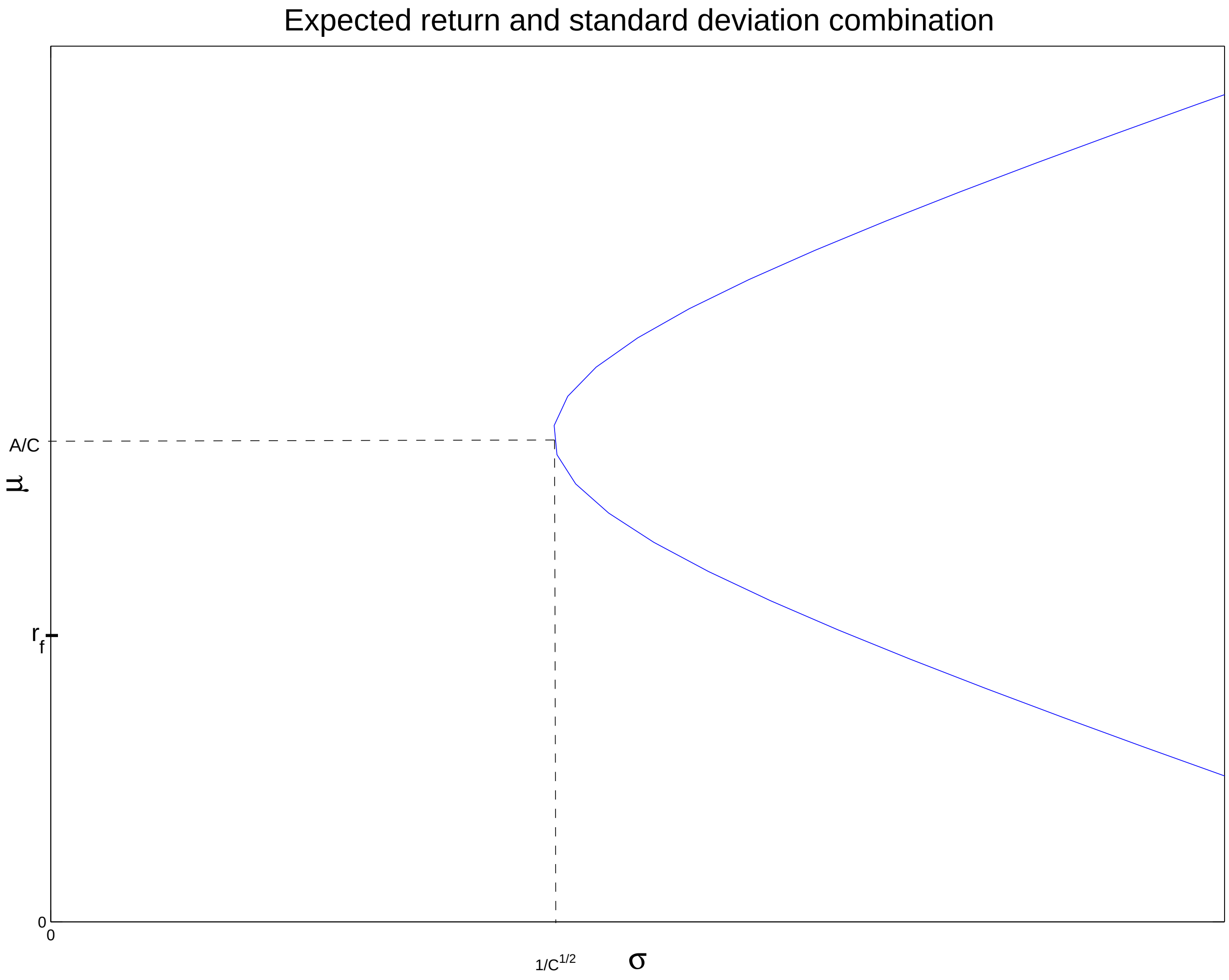
$$\sigma_p^2 = \frac{B - 2A\mu + C\mu^2}{D}$$



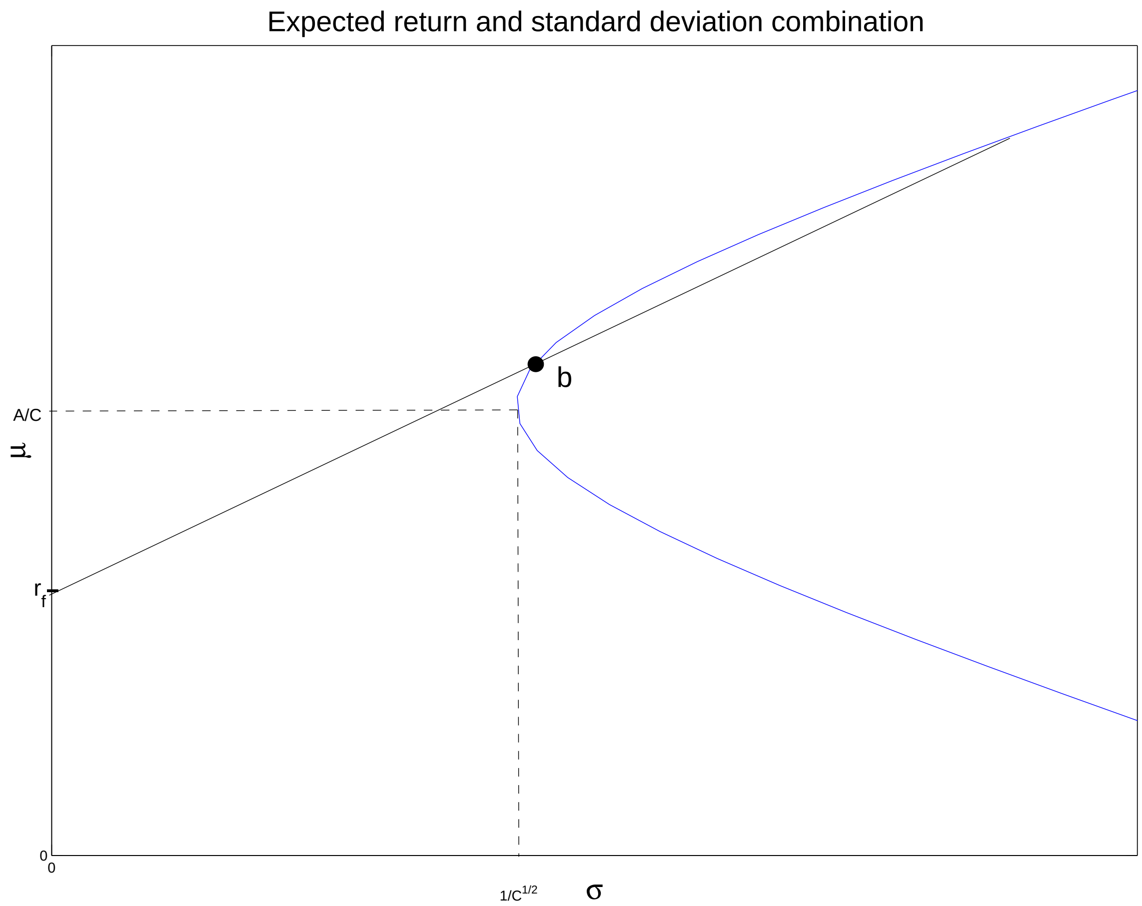
- In the $\mu - \sigma$ plane we get



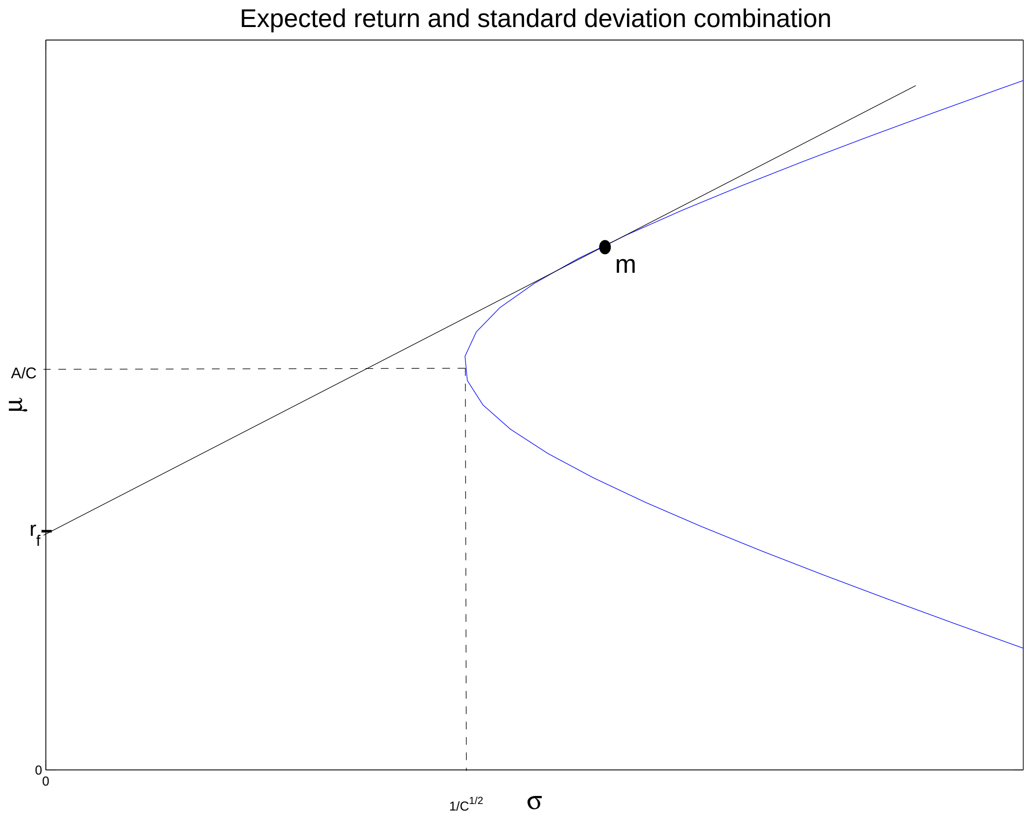
- It can be argued that in equilibrium we should have $r_f < A/C$



- If we create a portfolio by combining the risk-free asset with a portfolio b on the frontier, we could get the following combination of μ and σ



- The portfolio x_b would not be optimal. It is possible to get a higher μ for the same σ by switching from b to m (a portfolio that is tangent)



A combination of any other risky portfolio with the risk-free asset would give less μ for the same σ.

- It follows that everyone should choose a portfolio which falls on the $r_f - m$ line.
  - If you want higher pair (μ, σ), you put more weight on m.
  - If you want lower pair (μ, σ), you put more weight on risk-free asset.
  - The relative proportion of the risky assets should be the same regardless of where you are on the $r_f - m$ line. We refer to m as the market portfolio.
  - Your risk aversion will determine where on the $r_f - m$ line you are
    * Higher risk aversion $\Rightarrow$ close to r_f
    * Low risk aversion $\Rightarrow$ close to m or beyond m.

- What is the CAPM equation when we have a risk-free asset?
- consider the portfolio m (which is a portfolio on the frontier), then for a portfolio q

$$
\begin{aligned}
Cov(x_q, x_m) &= w_q' \Sigma w_m \\
&= w_q' \Sigma \left[\Sigma^{-1} (\bar{x} - i r_f) \frac{\mu_m - r_f}{H} \right] \\
&= w_q' (\bar{x} - i r_f) \frac{\mu_m - r_f}{H} \\
&= \frac{(\mu_q - r_f)(\mu_m - r_f)}{H}
\end{aligned}
$$

- But we also know from slide 28, when we take $\mu = \mu_m$, that

$$\begin{aligned} \sigma_m^2 &= \frac{(\mu_m - r_f)^2}{H} \\ \Rightarrow \quad \frac{\mu_m - r_f}{H} &= \frac{\sigma_m^2}{(\mu_m - r_f)} \end{aligned}$$

- We can next combine the last equation with the $Cov(x_q, x_m)$ equation on the previous slide to get

$$\begin{aligned} Cov(x_q, x_m) &= (\mu_q - r_f)\frac{\sigma_m^2}{\mu_m - r_f} \\ \mu_q - r_f &= \underbrace{\frac{Cov(x_q, x_m)}{\sigma_m^2}}_{=\beta_{qm}}(\mu_m - r_f) \end{aligned}$$

- Conclusion: the expected return of an asset/portfolio:
  - Does not depend on its variance.
  - Depends only on its covariance with the market.