11 Elements of asset pricing

11.1 Stochastic processes

1. Definition: A variable whose value evolves over time according to an established law of motion that is in some way dependent on a random variable.

   $\rightarrow$ The stochastic process itself is a random variable.

2. Examples

   - Autoregression of order p ($AR(p)$):

     $$X_t = \phi_0 + \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + \epsilon_t$$

     where ϵ_t is a "white noise":

     $$\begin{aligned} \epsilon_t &\sim N(0, \sigma^2) \\ E[\epsilon_t \epsilon_{t-h}] &= 0 \text{ if } h \neq 0 \end{aligned}$$

- Moving average of order q ($MA(q)$):

$$X_t = \phi_0 + \epsilon_t + \psi_1 \epsilon_{t-1} + \cdots + \psi_q \epsilon_{t-q}$$

- $ARMA(p,q)$:

$$X_t = \phi_0 + \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + \epsilon_t + \psi_1 \epsilon_{t-1} + \cdots + \psi_q \epsilon_{t-q}$$

- Random walk with drift:

$$\begin{aligned} X_t &= \phi_0 + X_{t-1} + \epsilon_t \\ \Leftrightarrow X_t - X_{t-1} &= \phi_0 + \epsilon_t \end{aligned}$$

- Continuous time
  - The foregoing examples are **discrete-time** stochastic processes.
  - For asset pricing (and many other things) it is more convenient to use **continuous-time** stochastic processes
  - We must define a continuous random variable and established its properties.

11.2 Markov process

1. A Markov process is one whose future values depend only on the process's current value, not any earlier values.

2. Examples

   - The random walk is Markov: $X_{t+1} = X_t + \epsilon_{t+1}$
   - The $AR(1)$ is Markov: $X_{t+1} = \phi_0 + \phi_1 X_t + \epsilon_{t+1}$
   - The $AR(2)$ is not Markov: $X_{t+1} = \phi_0 + \phi_1 X_t + \phi_2 X_{t-1} + \epsilon_{t+1}$

3. An implication: The probability distribution of a Markov process value in a given period is not dependent on the path followed by the process in the past.

4. Markov stock prices and market efficiency
   - Weak market efficiency
     - States that there are no unexploited profit opportunities because of market competition
     - All information contained in the past history of economic variables is contained in the current values
     - The current price of an asset contains all information present in the past prices of the asset.
   - Stock prices are Markov

11.3 Wiener processes (Brownian motion)

1. These are the continuous-time version of the random walk
2. Properties of a drift-less random walk $X_{t+1} = X_t + \epsilon_{t+1}$:
   - At time $t+1$ conditional on information up to time t:

$$\begin{aligned} E_t[X_{t+1}] &= X_t + E_t[\epsilon_{t+1}] \\ &= X_t \\ Var_t[X_{t+1}] &= Var_t[X_t + \epsilon_{t+1}] \\ &= \sigma^2 \end{aligned}$$

- At time $t+2$ conditional on information up to time t:

$$\begin{aligned}
E_t[X_{t+2}] &= E_t[X_{t+1} + \epsilon_{t+2}] \\
&= E_t[X_{t+1}] \\
&= X_t \\
Var_t[X_{t+2}] &= Var_t[X_{t+1} + \epsilon_{t+2}] \\
&= Var_t[X_t + \epsilon_{t+1} + \epsilon_{t+2}] \\
&= Var_t[\epsilon_{t+1}] + Var_t[\epsilon_{t+2}] \\
&= 2\sigma^2
\end{aligned}$$

- In general,

$$\begin{aligned}
E_t[X_{t+h}] &= X_t \\
Var_t[X_{t+h}] &= h\,\sigma^2
\end{aligned}$$

3. The important properties are
   - $E_t[X_{t+h} - X_t] = 0$: The expected change is equal to zero.
   - $Var_t[X_{t+h}] = h\,\sigma^2$: The variance is linear in time
   - $Std_t[X_{t+h}] = \sqrt{h}\,\sigma$: The standard deviation is proportional to the **square root** of time.

4. Wiener process

   (a) A random variable z is a Wiener process if:

   - Property 1: The change in z during a small interval of time Δt is $\Delta z = \epsilon\sqrt{\Delta t}$ where $\epsilon \sim N(0,1)$.
   - Property 2: The value of Δz for any two non-overlapping intervals are independent.

   (b) Property 1 $\Rightarrow z \sim N(0, \Delta t)$

   Property 2 $\Rightarrow z$ is Markov

(c) Consider the change in z over a long period of time: $z_T - z_0$.

- We can treat this as the sum of changes in z over short intervals of time of length Δt, where

$$N \equiv T/\Delta t$$

- Then,

$$z_T - z_0 = \sum_{i=1}^{N} \epsilon_i \sqrt{\Delta t}$$

- We take the limit as $\Delta t \to 0$ to get

$$dz = \epsilon \sqrt{dt}$$

5. Generalized Wiener process

   (a) $dx = a\,dt + b\,dz$

   (b) The term $a\,dt$ is the drift. If b were 0, then

$$\begin{array}{rcl} dx & = & a\,dt \\ \Leftrightarrow \dfrac{dx}{dt} & = & a \\ \Rightarrow x_t & = & x_0 + at \end{array}$$

   (c) The term $b\,dz$ adds noise or randomness to the path of x.

(d) We can write $dx = a\,dt + b\,\epsilon\sqrt{dt}$:

$$\begin{aligned} E[dx] &= a\,dt \\ Var[dx] &= b^2\,dt \end{aligned}$$

Also,

$$\begin{aligned} E[x_T - x_0] &= aT \\ Var[x_T - x_0] &= b^2 T \end{aligned}$$

6. Ito process:

$$\begin{aligned} dx &= a(x,t)\,dt + b(x,t)\,dz \\ &= a(x,t)\,dt + b(x,t)\,\epsilon\sqrt{dt} \end{aligned}$$

11.4 Stock price model

1. If there were no randomness, we would want the **growth rate** (rate of return) to be constant:

$$\begin{aligned} \frac{dS_t}{dt}\frac{1}{S_t} &= \mu \\ \Leftrightarrow dS_t &= \mu S_t\, dt \\ \Rightarrow S_T &= S_0 e^{\mu T} \qquad \text{exponential growth} \end{aligned}$$

2. To include randomness in the return, we use an Ito process

$$dS_t = \mu S_t\, dt + \sigma S_t\, dz$$

   where $a(S,t) = \mu S_t$ and $b(S,t) = \sigma S_t$.

3. The parameter μ and σ:
   - μ is the expected rate of return.
     - It should depend on underlying economic variables such as the interest rate and also the riskiness of the stock S.
     - We can ignore all this because the value of a derivative (which is what we want to price) generally is independent of μ.
   - σ is the volatility parameter.
     - It is very important in determining the value of most derivatives.

11.5 Ito's lemma

1. Derivatives are functions of stock prices.
   - The value of the derivative today is its discounted expected value (under risk neutrality) at the time of expiration.
   - The latter depends on the stock price at the time of expiration.
   - We have a model of the evolution of S, but what is the model for the evolution of the option's price?
   - If we denote the option price as a function F of S and time, $F(S,t)$, then we we want to know $dF(S,t)$.
   - To get it, we use Ito's lemma.

2. The lemma

   (a) Ito's lemma: If $dS = a\,dt + b\,dz$ and $F = F(S,t)$, then

   $$dF = \left(a\frac{\partial F}{\partial S} + \frac{\partial F}{\partial t} + \frac{1}{2}b^2\frac{\partial^2 F}{\partial S^2} \right) dt + b\frac{\partial F}{\partial S}dz$$

   (b) Derivation

      i. We can use the total differential to derive an approximation for the change in F in response to a small change in S:

      $$\Delta F \approx \frac{\partial F}{\partial S}\Delta S$$

      ii. Then by letting $\Delta S \to 0$, we get

      $$dF = \frac{\partial F}{\partial S}dS$$

iii. We could have started with a closer approximation by using a higher-order Taylor approximation

$$\Delta F \approx \frac{\partial F}{\partial S}\Delta S + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}(\Delta S)^2.$$

However, $(\Delta S)^2 \to 0$ faster than ΔS, so the second term becomes negligible with respect to the first as $\Delta S \to 0$.

iv. Taylor's approximation for a function of two variables $f(S,t)$:

$$\begin{aligned}
\Delta F &\approx \frac{\partial F}{\partial S}\Delta S + \frac{\partial F}{\partial t}\Delta t \\
&\quad + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}(\Delta S)^2 + \frac{1}{2}\frac{\partial^2 F}{\partial t^2}(\Delta t)^2 + \frac{\partial^2 F}{\partial S \partial t}(\Delta S)(\Delta t) \\
&\approx \frac{\partial F}{\partial S}\Delta S + \frac{\partial F}{\partial t}\Delta t \quad \text{for small } \Delta S \text{ and } \Delta t \\
&\rightarrow \frac{\partial F}{\partial S}dS + \frac{\partial F}{\partial t}dt \quad \text{as } \Delta S \rightarrow 0,\ \Delta t \rightarrow 0
\end{aligned}$$

v. However, when we apply Taylor's approximation to an Ito process, some of the second-order terms do not drop out. If

$$\begin{aligned} dS &= a\,dt + b\,dz \\ &= a\,dt + b\,\epsilon\sqrt{dt} \end{aligned}$$

where $a = a(S, t)$ and $b = b(S, t)$, then

$$
\begin{aligned}
\Delta F &\approx \frac{\partial F}{\partial S}\Delta S + \frac{\partial F}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}(\Delta S)^2 + \frac{1}{2}\frac{\partial^2 F}{\partial t^2}(\Delta t)^2 + \frac{\partial^2 F}{\partial S \partial t}(\Delta S)(\Delta t) \\
&= \frac{\partial F}{\partial S}\left[a\,\Delta t + b\,\epsilon\sqrt{\Delta t}\right] + \frac{\partial F}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}\left[a\,\Delta t + b\,\epsilon\sqrt{\Delta t}\right]^2 + \\
&\quad \frac{1}{2}\frac{\partial^2 f}{\partial t^2}(\Delta t)^2 + \frac{\partial^2 F}{\partial S \partial t}(a\,\Delta t + b\,\epsilon\sqrt{\Delta t})(\Delta t) \\
&= \frac{\partial F}{\partial S}\left[a\,\Delta t + b\,\epsilon\sqrt{\Delta t}\right] + \frac{\partial F}{\partial t}\Delta t \\
&\quad + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}\left[a^2(\Delta t)^2 + 2ab\epsilon(\Delta t)^{3/2} + b^2\epsilon^2\Delta t\right] + \\
&\quad \frac{1}{2}\frac{\partial^2 F}{\partial t^2}(\Delta t)^2 + \frac{\partial^2 F}{\partial S \partial t}\left[a(\Delta t)^2 + b\,\epsilon(\Delta t)^{3/2}\right]
\end{aligned}
$$

vi. As $\Delta t \to 0$, terms in Δt with powers greater than one will go to zero faster than Δt itself goes to zero. Therefore

$$\begin{aligned} dF &= \frac{\partial F}{\partial S}\left[adt + b\epsilon\sqrt{dt}\right] + \frac{\partial F}{\partial t}dt + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}\left[b^2\epsilon^2 dt\right] \\ &= \left[a\frac{\partial F}{\partial S} + \frac{\partial F}{\partial t} + \frac{1}{2}b^2\epsilon^2\frac{\partial^2 F}{\partial S^2}\right]dt + b\frac{\partial F}{\partial S}dz \end{aligned}$$

vii. There is one more refinement. Consider the term $\frac{1}{2}b^2\epsilon^2\frac{\partial^2 F}{\partial x^2}\Delta t$.

- We know that $E[\epsilon] = 0$ and $Var[\epsilon] = 1$.
- So

$$
\begin{aligned}
E[\epsilon^2\Delta t] &= E[\epsilon^2]\Delta t \\
&= \Delta t \\
Var[\epsilon^2\Delta t] &= E[(\epsilon^2\Delta t)^2] - E[\epsilon^2\Delta t]^2 \\
&= E[\epsilon^4(\Delta t)^2] - \left(E[\epsilon^2]\Delta t\right)^2 \\
&= \left(E[\epsilon^4] - 1\right)(\Delta t)^2
\end{aligned}
$$

- We see that the variance will go to zero faster than Δt as $\Delta t \to 0$.
- It follows that $\epsilon^2 \Delta t$ becomes non-stochastic as $\Delta t \to 0$.
- $\epsilon^2 \Delta t \to E[\epsilon^2 dt] = E[\epsilon^2] dt = dt$ as $\Delta t \to 0$.
- So we finally have

$$dF = \left(a \frac{\partial F}{\partial S} + \frac{\partial F}{\partial t} + \frac{1}{2} b^2 \frac{\partial^2 F}{\partial S^2} \right) dt + b \frac{\partial F}{\partial S} dz$$

which is Ito's lemma.

3. Application to financial markets

   (a) General result

   - The value of a financial derivative D is a function of
     - The value of the underlying asset S
     - Time (because we care about how much time is left until expiration).
   - So, $D = F(S, t)$.
   - Suppose S is an Ito process:

$$dS = a(S,t)dt + b(S,t)dz$$

   - Then the change in D as time passes is

$$dD = \left[a(S,t)\frac{\partial F}{\partial S} + \frac{\partial F}{\partial t} + \frac{1}{2}b(S,t)^2\frac{\partial^2 F}{\partial S^2}\right]dt + b(S,t)\frac{\partial F}{\partial S}dz$$

(b) Example: forward contract

- We have seen earlier that the value of a forward contract at time t with delivery at $T \geq t$ is

$$F_t = S_t e^{r(T-t)} \qquad \text{(assuming } r \text{ is constant)}$$

- **S process.** Suppose the process for S is

$$dS_t = \mu S_t dt + \sigma S_t dz_t$$

where we have $a(S,t) = \mu S_t$ and $b(S,t) = \sigma S_t$.

- **F process.**

$$dF_t = \left(\mu S_t \frac{\partial F}{\partial S} + \frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 F}{\partial S^2} \right) dt + \sigma S_t \frac{\partial F}{\partial S} dz_t$$

In this example,

$$
\begin{aligned}
\frac{\partial F}{\partial S} &= e^{r(T-t)} \\
\frac{\partial F}{\partial t} &= -rS_t e^{r(T-t)} \\
\frac{\partial^2 F}{\partial S^2} &= 0
\end{aligned}
$$

Therefore

$$
\begin{aligned}
dF_t &= \left(\mu S_t e^{r(T-t)} - rS_t e^{r(T-t)}\right) dt + \sigma S_t e^{r(T-t)} dz_t \\
&= (\mu F_t - rF_t)\, dt + \sigma F_t dz_t \\
&= (\mu - r)\, F_t dt + \sigma F_t dz_t
\end{aligned}
$$

Note that

$$\begin{aligned} E[dF_t] &= (\mu - r)F_t dt \\ \Rightarrow \frac{E[dF_t]}{dt}\frac{1}{F_t} &= \mu - r \end{aligned}$$

which is the excess return of S over the risk-free rate r.

- Suppose we don't know the value of F_t. We can use the equation for dF_t to find F_t. We solve an equation closely related to the foregoing equation, subject to known boundary conditions. We discuss this method later.

11.6 Log-normal property

- Let $F(S) = \ln S$ and

$$dS = \mu S dt + \sigma S_t dz_t$$

- Then

$$\begin{aligned} \frac{\partial F}{\partial S} &= \frac{1}{S} \\ \frac{\partial^2 F}{\partial S^2} &= -\frac{1}{S^2} \\ \frac{\partial F}{\partial t} &= 0 \end{aligned}$$

- $\Rightarrow dF = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dz$.
- This is a generalized Wiener process because μ and σ are constants.

- Integrating,

$$\begin{aligned} \ln S_T - \ln S_t &\sim N\left(\left(\mu - \frac{\sigma^2}{2}\right)(T-t), \sigma^2(T-t)\right) \\ \ln S_T &\sim N\left(\ln S_t + \left(\mu - \frac{\sigma^2}{2}\right)(T-t), \sigma^2(T-t)\right) \end{aligned}$$

- The reason the growth rate is $\mu - \frac{\sigma^2}{2}$ instead of just μ is Jensen's inequality ($\ln$ is a concave function).

11.7 Rates of return

1. **Expected instantaneous rate of return.**

   - The geometric Wiener process

$$dS_t = \mu S_t dt + \sigma S_t dZ_t$$

   - has an expected change of

$$\begin{aligned} E[dS_t] &= \mu S_t dt \\ \Rightarrow \frac{E[dS_t]}{dt}\frac{1}{S_t} &= \mu \end{aligned}$$

2. **Expected continuously compounded rate of return between time 0 and T.**

   - Let η be this compounded rate:

$$\begin{aligned} S_T &= S_0 e^{\eta T} \\ \eta &= \frac{1}{T} \ln\left(\frac{S_T}{S_0}\right) \\ \Rightarrow \eta &\sim N\left(\mu - \frac{\sigma^2}{2}, \frac{\sigma^2}{T}\right) \qquad \text{(using previous result)} \end{aligned}$$

   - We can see that

$$\begin{aligned} E[\eta] &= \mu - \frac{\sigma^2}{2} \qquad \text{(mean growth rate for } \ln S\text{)} \\ &\neq \mu \qquad \text{(mean growth rate for } S\text{)} \end{aligned}$$

   - This is again a consequence of Jensen's inequality.

11.8 Multivariate version of Ito's lemma

1. Two-dimensional case

   - Suppose two Ito processes $S_{1,t}$ and $S_{2,t}$:

$$\begin{bmatrix} dS_{1,t} \\ dS_{2,t} \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} dt + \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \begin{bmatrix} dW_{1,t} \\ dW_{2,t} \end{bmatrix}$$

     where

     – a_i, σ_{ij} can be functions of S_i and t.

     – $E[dW_{1,t}dW_{2,t}] = \rho dt$.

   - If we have a function F of S_1 and S_2, what is dF?

- Ito's lemma – bivariate form

$$\begin{aligned} dF &= F_t dt + F_{S_1} dS_1 + F_{S_2} dS_2 \\ &\quad + \frac{1}{2}\left[F_{S_1 S_1} {dS_1}^2 + 2F_{S_1 S_2} dS_1 dS_2 + F_{S_2 S_2} {dS_2}^2\right] \end{aligned}$$

- We have:

$$\begin{aligned} {dS_1}^2 &= [a_1 dt + \sigma_{11} dW_{1,t} + \sigma_{12} dW_{2,t}]^2 \\ &= a_1^2 dt^2 + \sigma_{11}^2 dW_{1,t}^2 + \sigma_{12}^2 dW_{2,t}^2 + 2a_1\sigma_{11} dt\, dW_{1,t} \\ &\quad +2a_1\sigma_{12} dt\, dW_{2,t} + 2\sigma_{11}\sigma_{12}\, dW_{1,t}\, dW_{2,t} \\ &= 0 + \sigma_{11}^2 dt + \sigma_{12} dt + 0 + 0 + 2\sigma_{11}\sigma_{12}\rho\, dt \qquad \text{as } \Delta t \to 0 \\ &= \left(\sigma_{11}^2 + \sigma_{12}^2 + 2\sigma_{11}\sigma_{12} + \rho\right) dt \end{aligned}$$

$$ {dS_2}^2 = (\sigma_{22}^2 + \sigma_{21}^2 + 2\sigma_{21}\sigma_{22}\rho) dt $$

$$ dS_1 dS_2 = \left[\sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22} + \rho\left(\sigma_{11}\sigma_{22} + \sigma_{12}\sigma_{21}\right)\right] dt $$

- Substituting into the expression for dF and rearranging terms gives

$$\begin{aligned} dF &= \Big\{ F_t + a_1 F_{S_1} + a_2 F_{S_2} + \frac{1}{2} F_{S_1 S_1} \left(\sigma_{11}^2 + \sigma_{12}^2 + 2\rho\sigma_{11}\sigma_{12} \right) \\ &\quad + F_{S_1 S_2} \left[\sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22} + \rho \left(\sigma_{11}\sigma_{22} + \sigma_{12}\sigma_{21} \right) \right] \\ &\quad + \frac{1}{2} F_{S_2 S_2} \left(\sigma_{21}^2 + \sigma_{22}^2 + 2\rho\sigma_{21}\sigma_{22} \right) \Big\} dt \\ &+ \left(\sigma_{11} F_{S_1} + \sigma_{21} F_{S_2} \right) dW_{1,t} + \left(\sigma_{12} F_{S_1} + \sigma_{22} F_{S_2} \right) dW_{2,t} \end{aligned}$$

2. Financial derivative application

   - Consider an option written on a bond. The value of the bond depends on the entire array of interest rates, that is, on the yield curve.
   - Suppose the entire yield curve can be be adequately captured by the behavior of two interest rates:
     - r: short rate
     - R: long rate

     $\Rightarrow$ Derivative also depends on r and R.
   - Suppose

$$\begin{bmatrix} dr \\ dR \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} dt + \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \begin{bmatrix} dW_{1,t} \\ dW_{2,t} \end{bmatrix}$$

     with $E[dW_{1,t}dW_{2,t}] = 0$, i.e. $\rho = 0$.

- Note that,even though dW_1 and dW_2 are uncorrelated, dr and dR are correlated because they each depend on both Wiener processes:

$$Cov(dr, dR) = (\sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22})dt$$

- The behavior of the option's price is given by

$$\begin{aligned} dF &= \Big\{F_t + a_1 F_r + a_2 F_R + \frac{1}{2}F_{rr}\left(\sigma_{11}^2 + \sigma_{12}^2\right) \\ &\quad + F_{rR}\left[\sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22}\right] + \frac{1}{2}F_{RR}\left(\sigma_{21}^2 + \sigma_{22}^2\right)\Big\}dt \\ &\quad + \left(\sigma_{11}F_r + \sigma_{21}F_R\right)dW_{1,t} + \left(\sigma_{12}F_r + \sigma_{22}F_R\right)dW_{2,t} \end{aligned}$$