5 Forward and Futures Prices

5.1 Simple asset (no income, no costs, etc.)

- Basic formula

$$F_0 = S_0 e^{rT}$$

where

T	:	Time until delivery date
S_0	:	price of the underlying asset
F_0	:	forward or futures price today
r	:	risk-free rate of interest per annum (continuous compounding)

- Equivalently

$$S_0 = F_0 e^{-rT}.$$

- Market dynamics
  - If $F_0 > S_0 e^{rT}$:
    1. Buy the asset now (go long in the asset) at S_0.
    2. Short forward contracts on the asset (promise to sell the asset in the future at F_0).
    3. Profit at delivery date $= F_0 - S_0 e^{rT}$.
  - If $F_0 < S_0 e^{rT}$:
    1. Short the asset now (borrow the asset and sell it now for S_0).
    2. Go long on forward contracts on the asset (promise to buy the asset in the future for F_0).
    3. Profit $= S_0 e^{rT} - F_0$.

- Such activities continue “until profit = 0”:
  - S_0 is driven up (when $F_0 > S_0 e^{rT}$) or down (when $F_0 < S_0 e^{rT}$) because today arbitrageurs are trying to buy or sell.
  - F_0 is driven down (when $F_0 > S_0 e^{rT}$) or up (when $F_0 < S_0 e^{rT}$) because arbitrageurs are offering to sell or buy in the future.

- Convergence

$$S_t = F_t e^{-r(T-t)} \quad \Longrightarrow \quad S_T = F_T.$$

5.2 Modifications

5.2.1 Known income

Let I_0 be the present value of the future income stream, then

$$\begin{aligned} F_0 &= (S_0 - I_0)e^{rT} \\ \Rightarrow \qquad S_0 &= F_0 e^{-rT} + I_0. \end{aligned}$$

Intuition:

- Let $T = 0$, then F is the spot price less income.
- At time T, I will receive the asset minus the income stream.

5.2.2 Known yield

- Income is not fixed. Instead it is a variable percentage of the asset's price at the time the income is paid (e.g., a stock index).

$$\begin{aligned} r &= \text{interest rate (assumed constant)} \\ q_t &= \text{asset's yield} \\ p_t &= \text{wealth at time } t \end{aligned}$$

- There are two investment strategies:
  1. Put p into the risk-free asset and earn r. At any time t, the rate of earning is
  $$rp_t.$$
  2. Put p into the other asset and earn q plus any capital gain, so income at any time t is
  $$q_t p_t + \frac{dp_t}{dt}.$$

- Arbitrage guarantees that the two strategies must have the same payoff:

$$
\begin{aligned}
r p_t &= q_t p_t + \frac{dp_t}{dt} \\
\Rightarrow \qquad \frac{dp_t}{dt} &- (r - q_t) p_t = 0
\end{aligned}
$$

which is a homogeneous ODE. The integrating factor is

$$e^{-rt+\int_0^t q_s ds}.$$

So the solution to the ODE is

$$
\begin{aligned}
\int_0^T \left[\frac{d}{dt} \left(p_t e^{-rt+\int_0^t q_s ds} \right) \right] dt &= 0 \\
p_T e^{-rT+\int_0^T q_s ds} - p_0 &= 0.
\end{aligned}
$$

which gives

$$\begin{aligned} p_T &= p_0 e^{rT - \int_0^T q_s ds} \\ &= p_0 e^{rT - T(\frac{1}{T}\int_0^T q_s ds)} \\ &= p_0 e^{(r-\bar{q})T} \end{aligned}$$

where $\bar{q} \equiv \frac{1}{T}\int_0^T q_s ds$ is the average yield on the asset.

- If we now note that

$$\begin{aligned} p_0 &= \text{spot price } S_0 \\ p_T &= \text{period-0 forward price } F_0 \end{aligned}$$

we get

$$F_0 = S_0 e^{(r-\bar{q})T}.$$

- Can also just specify the (net) opportunity cost rate as $r - \bar{q}$.

5.3 Value of a forward contract

$$f = \text{value of an existing forward contract today}$$
$$K = \text{delivery price in that contract}$$

- Basic formulae
  - Long contract: $f = (F_0 - K)e^{-rT}$
  - Short contract: $f = (K - F_0)e^{-rT}$

- Elaborations:
  - Simple asset:

$$F_0 = S_0 e^{rT}$$
$$\Rightarrow \qquad f = S_0 - Ke^{-rT} \qquad \text{(long contract)}$$

  - Known income:

$$F_0 = (S_0 - I_0)e^{rT} \qquad (I_0 \equiv \text{present value of income stream})$$
$$\Rightarrow \qquad f = S_0 - I_0 - Ke^{-rT} \quad \text{(long contract)}$$

  - Known constant yield:

$$F_0 = S_0 e^{(r-q)T}$$
$$\Rightarrow \qquad f = S_0 e^{-qT} - Ke^{-rT} \qquad \text{(long contract)}$$

5.4 Forward contract on foreign currency

$$F_0 = S_0 e^{(r - r_F)T}$$

where $r_F =$ foreign country's risk-free rate.

We can think of the foreign currency as an asset paying a known yield of r_F.

5.5 Forward on commodities

- Formula: $F_0 = (S_0 + U)e^{rT}$ where $U \equiv$ present value of all the storage costs (intuition: let $T = 0$, then $S = F - U$).
- Alternative possibility: storage costs proportional to the price of the commodity:

  $$F_0 = S_0 e^{(r+u)T}$$

  where $u \equiv$ storage cost rate.

  Nota that u is, in effect, a negative income and enters that way in the formula.

- Concumption commodities and convinience yield:
  1. Suppose $F_0 > (S_0 + U)e^{rT}$

     To take advantage of this situation:

     (a) Borrow $S_0 + U$ at rate r
     (b) Buy one unit of the commodity and pay storage cost
     (c) Short a forward contract on one unit of the commidity.
     (d) Realize a profit of $F_0 - (S_0 + U)e^{rT}$ at time T.

     There is no problem implementing this strategy. Doing so will tend to make F_0 fall and S_0 rise, eliminating the arbitrage opportunity.

2. Suppose $F_0 < (S_0 + U)e^{rT}$

   For an investment asset (for which stocks exist), use the following strategy:

   (a) Sell the commodity now; avoid paying the storage costs.
   (b) Invest proceeds at risk free rate r.
   (c) Go long in a forward contract
   (d) Receive a (riskless) profit of $(S_0 + U)e^{rT} - F_0$.

   As before, this kind of arbitrage causes S_0 and F_0 to change to restore the equality $F_0 = (S_0 + U)e^{rT}$.

3. A problem with consumption goods: there may be no stocks available for borrowing.
   - Stocks may exist; they just are not available for borrowing because their owners are holding them to consume them, not to make an investment profit from them.
   - For such consumption goods, the inequality $F_0 < (S_0 + U)e^{rT}$ may persist.
   - Users of the consumption good feel that ownership of a commodity provides benefits that are not available from holding a forward or futures contract instead.
   - These extra benefits give a flow of services accruing at a rate called the **convenience yield**, defined as the rate y that makes the following equation hold:

$$\begin{aligned} F_0 e^{yT} &= (S_0 + U)e^{rT} \qquad \text{[or } S_0 e^{(r+u)T} \text{ if appropriate]} \\ \Rightarrow \qquad F_0 &= (S_0 + U)e^{(r-y)T}. \end{aligned}$$

5.6 Futures prices and expected spot prices

Is it true that $F_0 = E[S_T]$? Generally , **no**:

- Compensation for risk as an insurance premium
  1. Hedgers avoid risk by shifting it to speculators, who assume the risk.
  2. To be compensated for bearing the risk, speculators demand a premium
  3. The premium is paid by having $E[S_T] \neq F_0$.
     - If hedgers are short and speculators are long (will buy), $F_0 < E[S_T]$ ("normal backwardation") so that speculators can expect a profit of $E[S_T] - F_0$ and hedgers expect a loss of the same amount.
     - If hedgers are long and speculators short, $F_0 > E[S_T]$ ("contango") giving speculators an expected profit of $F_0 - E[S_T]$.
     - Hedgers' expected loss in effect is the insurance premium paid to speculators.

- Systematic and nonsystematic risks

  1. Capital asset pricing model (more details later)
     - Systematic risk — elements of risk common to all assets, arising from economy-wide factors.
     - Nonsystematic risks — elements of risk specific to an asset.
     - Nonsystematic risk can be nullified by holding a diversified portfolio that averages out all nonsystematic risks.
     - Systematic risk cannot be avoided.
       In particular, that part of an asset's risk that is correlated with the stock market is systematic risk.

2. Risk in a futures position
   - Long position
     (a) Put present value of futures price into safe asset and take long futures position.
     Saving now $= F_0 e^{-rT}$.
     (b) At time T, buy the asset for F_0 and re-sell it for S_T.
     (c) The expected present value of the sale price is the expected sale price $E[S_T]$ discounted by the **risk-adjusted** discount rate k, $E[S_T]e^{-kT}$.
     [We will discuss k in more detail later in the course. For now it is sufficient to think of it as the rate of return required to induce investors to hold asset.]

– Arbitrage guarantees that all investment opportunities have 0 net present value:

$$E[S_T]e^{-kT} - F_0 e^{-rT} = 0$$

From which we get

$$\begin{aligned} F_0 &= E[S_T]e^{(r-k)T} \\ &< E[S_T] \qquad \text{if} \quad k > r \end{aligned}$$

The excess of k over r represents the systematic risk asociated with the asset.
[We will see more on this later, but $k > r$, $k = r$, $k < r$ as the correlation of S_T with stock market is > 0, $= 0$, < 0 respectively.]

– Short position

(a) Borrow $E[S_T]e^{-kT}$, buy the asset, and take a short futures position.

(b) At time T, sell the asset for F_0, and use the proceeds to repay the loan plus interest $E[S_T]e^{(r-k)T}$.

(c) Zero profit: $F_0 = E[S_T]e^{(r-k)T}$ as before.