14 Greeks Letters and Hedging

14.1 Illustration

- We consider the following example through out this section.
- A financial institution sold for $300,000 a European call option on 100,000 shares of a non-dividend paying stock. We assume that
  - The stock price is $S_0 = 49$
  - The strike price is $K = 50$
  - The risk-free interest rate is $r = 5\%$ per annum
  - The stock price volatility is $\sigma = 20\%$ per annum
  - The time to maturity is 20 weeks ($T = 0.3846$ years)
  - The expected return from the stock is 13% per annum.

- The Black-Scholes price for this option is about $237,000.

$$
\begin{aligned}
d_1 &= \frac{\ln\left(\frac{49}{50}\right) + \left(0.05 + \frac{0.2^2}{2}\right)0.3846}{0.2\sqrt{0.3846}} \\
&= 0.0542 \\
d_2 &= \frac{\ln\left(\frac{49}{50}\right) + \left(0.05 - \frac{0.2^2}{2}\right)0.3846}{0.2\sqrt{0.3846}} \\
&= -0.0699 \\
c &= 49N(0.0542) - 50e^{-0.05\times 0.3846}N(-0.0699) \\
&= 2.3745
\end{aligned}
$$

- When buying/selling options you can have a:
  - **Naked position:** you buy/sell the option and do nothing else.
    - In our example, ideal if the stock price falls below $50. You keep the $300,000 and don't pay anything.
    - But if the stock price goes above $50, you then have to pay. For example, if the price in 20 weeks is $60, then you have to pay $1 million.
  - **Covered position:** you buy (short) shares on the days you sell (buy) a call option.
    - In our example, the institution would buy 100,000 shares.
    - If the price in 20 weeks is above $50, then the losses form the option are offset by the gains on the stock.
    - But if the price in 20 weeks is below $49, then you lose money on your long stock position.

- We see that naked position and covered position are not perfect.
- There is uncertainty about the cost. It can be as little as zero, or $1 million, or even more.
- With a perfect hedge, the discounted expected cost to the institution should be about $237,000.

14.2 Stop-loss strategy for hedging

- From the results above, we see that when selling a call option, you would like to have a covered position when the price is above K and you would like to have a naked position when the price is below K.
- Stop-loss strategy: You buy the stock when the price goes above K and you sell it when the price falls below K. You might buy and sell many times over the life of the option.
- Does not really work in practice:
  - You pay K for the stock before maturity of the option, which once discounted is more than K at maturity.
  - Bid-ask spread: when we say that the price is S_t, it means that you buy at $S_t + \epsilon$ (bid) and you sell at $S_t - \epsilon$ (ask). And that does not even include possible transaction costs.

14.3 Delta hedging

- We introduced the delta of an option, Δ, previously.
- Δ is the rate of change of the option price with respect to the price of the underlying asset:

$$\Delta = \frac{\partial c}{\partial S}$$

- If the delta of an option is 0.4, then if the stock price changes by a small amount, the option price changes by about 40% of the small amount.
- From Black-Scholes, we know that the value of a portfolio composed of -1 call option and Δ shares of the stock will not change for small change in the asset price.

- Delta of a European call option: $\Delta = N(d_1)$
- Delta of a European put option: $\Delta = N(d_1) - 1$
- In our example, if we delta-hedge, the cost of the call option should be $237,000.
- We have two examples in Hull. One where the asset price is above the strike price at maturity and one where the asset price is below.

Week	S_t	Δ	Shares purchased	Costs of shares pur. ($000)	Cum. cost inc. int. ($000)	Interest cost ($000)
0	49.00	0.522	52,200	2,557.8	2.557.8	2.5
1	48.12	0.458	(6,400)	(308.0)	2,252.3	2.2
2	47.37	0.400	(5,800)	(274.7)	1,979.8	1.8
3	50.25	0.596	19,600	984.9	2996.6	2.9
⋮						
18	54.62	0.990	1,200	65.5	5,197.3	5.0
19	55.87	1.000	1,000	55.9	5,258.2	5.1
20	57.25	1.000	0	0.0	5,263.3	

- Week 0: Since $S_t = 49.00$, $\Delta = 0.522$. Hence, we must borrow $2,557,800 to buy 52,000 shares of stock. The interest rate is 5%, so the interest payment on the loan that we will have to pay next week is $2,500.
- Week 1: The price goes down to $48.12. Delta is now $\Delta = 0.458$ (maturity is now 19 weeks). We can resell 6,400 shares [(0.458 - 0.522) $\times$ 100,000] at $48.12 a share for $308,000. The cumulative cost is $2,557.8 + 2.5 - 308 = 2,252.3$ thousand dollars. Next week, the interest payment on the loan will be $2,200.

- As we get closer to week 20, we see that at maturity the stock price will probably be above $50 and the call option will be exercised. The delta hedge increases to 1. The cumulative cost is $5,263,300, we receive 5 million dollars from whoever exercise the call. Hence, the total cost of the call option is $263,300.
- The total cost is not $237,000 because we rebalance the portfolio every week instead of continuously.
- In practice, we don't want to rebalance the portfolio continuously (or very often) because the transaction costs would be to big.

- The delta of a portfolio of options dependent on a single asset S is $\frac{\partial \Pi}{\partial S}$ where Π is the value of the portfolio.
- If the portfolio is composed of a quantity ω_i of of option i, then the delta of the portfolio is

$$\Delta = \sum_{i=1}^{n} \omega_i \Delta_i$$

where Δ_i is the delta of the ith option.

14.4 Theta

- The theta of a portfolio of options, Θ, is the rate of change of the value of the portfolio with respect to the passage of time with all else remaining constant.
- For a European call option on a non-dividend-paying stock, from the Black-Scholes formula we get

$$\Theta = \frac{\partial c}{\partial T} = -\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}N(d_2)$$

where

$$N'(d_1) = \frac{1}{\sqrt{2\pi}}e^{-d_1^2/2}$$

- For a European put option on a non-dividend-paying stock,

$$\Theta = -\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}} + rKe^{-rT}N(-d_2)$$

where

$$N'(d_1) = \frac{1}{\sqrt{2\pi}}e^{-d_1^2/2}$$

- Θ is usually negative for an option. As time to maturity decreases with all else remaining constant, the option tends to become less valuable.

- Do we need to hedge against Θ? No! Passage of time is deterministic. Hence, there is no uncertainty.

- Nonetheless, many see Θ as a useful descriptive statistic for a portfolio.

14.5 Gamma

- The gamma of a portfolio of options on an underlying asset, Γ, is the rate of change of the portfolio's delta with respect to the price of the underlying asset:

$$\Gamma = \frac{\partial^2 \Pi}{\partial S^2}$$

- If Γ is small, then Δ changes slowly when the asset price changes. It follows that this portfolio could be rebalance infrequently.

- If Γ is big, then Δ changes quickly when the asset price changes. It follows that this portfolio would need to be rebalanced frequently.

- Assuming that the asset price follows a geometric brownian motion and that the volatility is constant, the value of the portfolio, Π is a function of S and t.

- A Taylor series expansion of Π gives

$$\Delta\Pi = \frac{\partial \Pi}{\partial S}\Delta S + \frac{\partial \Pi}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 \Pi}{\partial S^2}\Delta S^2 + \frac{1}{2}\frac{\partial^2 \Pi}{\partial t^2}\Delta t^2 + \frac{\partial^2 \Pi}{\partial S \partial t}\Delta S \Delta t + \cdots$$

- Terms other than the first three on the right-hand side are of order higher than Δt. We can drop them since they are negligible:

$$\Delta\Pi = \frac{\partial \Pi}{\partial S}\Delta S + \frac{\partial \Pi}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 \Pi}{\partial S^2}\Delta S^2$$

- Using our notation

$$\Delta\Pi = \boldsymbol{\Delta}\Delta S + \boldsymbol{\Theta}\Delta t + \frac{1}{2}\boldsymbol{\Gamma}\Delta S^2$$

- We can easily make the portfolio delta neutral by buying the right number of stock shares. Doing so, we get

$$\Delta\Pi = \Theta\Delta t + \frac{1}{2}\Gamma\Delta S^2$$

- We would like to make the portfolio gamma neutral, i.e. immunized against bigger changes in the stock price.
- This can't be done by changing the number of shares we have in the portfolio. The value of the portfolio is a linear function of the asset price and the second derivative of a linear function is zero.
- We need a position in a derivative that is not linearly dependent on the underlying asset.

- Suppose a delta-neutral portfolio has a gamma equal to Γ.
- Suppose a traded option has a gamma equal to Γ_T (T for "traded", not maturity of the option).
- If we add ω_T units of the traded options to the portfolio, the gamma of the portfolio is

$$\omega_T \Gamma_T + \Gamma$$

- Hence, if we include $\omega_T = -\Gamma/\Gamma_T$ traded option to our portfolio, it will become gamma-neutral.
- Including additional options to our portfolio will likely change its delta. We might need to buy/sell units of the underlying assets to keep the portfolio delta neutral.

- delta neutral: protection against small changes in the underlying asset price.
- gamma neutral: protection against larger price movements.
- For a European call or put option on a non-dividend paying stock, gamma is given by

$$\Gamma = \frac{N'(d_1)}{S_0 \sigma \sqrt{T}}$$

14.6 Vega

- So far we assumed that the volatility σ was constant.
- In practice, it changes over time.
- This means that the value of a derivative can change because of a movement in volatility.
- The vega of a portfolio of derivatives, ν, is the rate of change of the value of the portfolio with respect to the volatility of the underlying asset:

$$\nu = \frac{\partial \Pi}{\partial \sigma}$$

- The vega of a position in the underlying asset is zero.
- The vega of a portfolio can be changed by adding a position in a traded option.
- If ν is the vega of the portfolio and ν_T is the vega of a traded option, a position $-\nu/\nu_T$ in the traded option makes the portfolio instantaneously vega neutral.
- A portfolio that is gamma neutral will not in general be vega neutral, and vice versa.
- If a hedger requires a portfolio to be both gamma neutral and vega neutral, at least two different traded options dependent on the underlying asset must be used.
- For a European call or put option on a non-dividend-paying stock,

$$\nu = S_0\sqrt{T}N'(d_1)$$

14.7 Rho

- The rho of a portfolio of options is the rate of change of the value of the portfolio with respect to the interest rate:

$$rho = \frac{\partial \Pi}{\partial r}$$

- For a European call option on a non-dividend-paying stock,

$$rho = KTe^{-rT}N(d_2)$$

14.8 Hedging in practice

- In an ideal world, a trader would like to rebalance his portfolio to get zero delta, zero gamma, zero, vega, ...
- In practice, it can't be done.
- They usually rebalance the portfolio daily to get zero delta (mainly trading the underlying asset).
- Zero gamma and zero vega are harder to achieve. The option market is not as deep as the market for the underlying asset. It might be expensive to get the volumes needed to get zero gamma and zero vega. Gamma and vega are monitored and corrective actions are taken when they get too large.