6 Hedging Using Futures

6.1 Types of hedges using futures

Two types of hedge: short and long.

1. **Short hedge**

- Take a short hedge position in the futures market.
- Appropriate when someone expects to sell an asset he already owns and wants to guarantee the price.
- Example: I'm selling 1 million barrels of crude oil. The spot price for a barrel is $19 per barrel and the 3-month futures price is $18.75 per barrel.
  - Spot price in three months proves to be $17.50: I gain $18.75-$17.50=$1.25 per barrel from the futures but I'm selling the oil for $1.25 less per barrel. I end up getting $18.75 per barrel.
  - Spot price in three months proves to be $19.50: I lose $19.50-$18.75=$0.75 per barrel from the futures but I'm selling the oil for $0.75 more per barrel. I end up getting $18.75 per barrel.

2. **Long hedge**
   - Take a long position in the futures market.
   - Appropriate for someone who expects to buy an asset and wants to guarantee the price.

6.2 Basis and basis risk

- **Perfect hedge does not always exist**
  - The asset we are trying to hedge may not be exactly the same as the asset underlying the futures.
  - The time at which we sell the asset (which could be random) might not be exactly be the same as the delivery date of the futures.

- **Basis**

1. Basis $\equiv$ spot price of asset to be hedged - futures price of asset being used for the hedge

$$b_t = S_t - F_t$$

If the asset being hedged and used for the hedge are the same, then the basis will be zero at the expiration of the futures contract.

2. Intuition of why it is useful:
   - Consider a short hedger who will sell an asset at time T and who takes a short futures position at $t_0 < T$ for delivery at some time $t_1 \geq T$.
   - Income at T is:

     (i) price of asset at time T: S_T

     (ii) profit on futures position:

$$(F_{t_0,t_1} - F_{T,t_1})e^{-r(t_1-T)} \approx F_{t_0,t_1} - F_{T,t_1}$$

     if $t_1 - T$ is small, as is usually the case.

     $\Rightarrow$ Total income at T

$$\begin{aligned} &\approx S_T + (F_{t_0,t_1} - F_{T,t_1}) \\ &= F_{t_0,t_1} + (S_T - F_{T,t_1}) \\ &= F_{t_0,t_1} + b_T \end{aligned}$$

   - The basis b_T collects all the uncertain terms and so collects all the risk into one number.

3. Same formula for basis applies to a long hedge
   - Consider someone who knows he will buy an asset at T and takes a long position at t_0
   - Total payment at T is
     (i) spot price S_T
     (ii) loss on hedge $F_{t_0,t_1} - F_{T,t_1}$ (approximately, as before)
     $\Rightarrow$ total payment

$$\begin{aligned} &= S_T + (F_{t_0,t_1} - F_{T,t_1}) \\ &= F_{t_0} + b_T \end{aligned}$$

     as before. Note that $b_T \equiv 0$ if the futures contract expires at T because then $S_T = F_T$.

- **Basis risk**
  - If b_T were known at t_0, we would have a perfect hedge (because if b_T is known, then b_T is fixed. This means that S_T and F_T must always change by equal amounts, leaving income unchanged).
  - When b_T is unknown at time t_0, the uncertainty about the period T income, captured by the uncertainty about the value of b_T, hence called **basis risk**.

- **Another source of risk: different assets held and contracted**
  - The asset in the futures contract may not be the same as the asset
  - Define $S^\star \equiv$ price of asset in futures contract
  - Total income/payment becomes

$$S_T + F_{t_0,t_1} - F_{T,t_1} = F_{t_0,t_1} + (S^\star_T - F_{T,t_1}) + (S_T - S^\star_T)$$

where

$S^\star_T - F_{T,t_1}$	=	basis if the asset being hedged were the same as the asset in the futures contract.
$S_T - S^\star_T$	=	difference in price between the two assets.

6.3 Minimum variance hedge ratio

- **Hedge ratio**
  - The size of the position in futures contracts relative to the exposure (i.e., the value of the asset being hedged).
  - More simply, the number of units of asset in futures contracts per unit of asset being hedged.

- **Riskiness of a short hedger's portfolio**

  1. Value of portfolio in arbitrary time period t
     - $N_A \equiv$ number of units of asset being hedged.
     - $h \equiv$ hedge ratio (ratio of the size of the position taken in futures contracts to the size of the exposure).
     - Then value of the portfolio is the sum of:

       (i) $N_A S_t =$ spot value of asset holding

       (ii) $N_A h(K - F_{t,T})e^{-r(T-t)} =$ value of the futures contracts owned where T is expiration date of futures contract

       $\Rightarrow$ total portfolio value $= N_A[S_t + h(K - F_{t,T})e^{-r(T-t)}]$.

2. Value at time when delivery of asset is made
   - Denote this time by t_1, with $t_1 \leq T$
   - Suppose the futures contract was written in period $t_0 < t_1$, so that $K = F_{t_0}$.
   - Value of portfolio in period t_1 is

$$N_A[S_{t_1} + h(F_{t_0,T} - F_{t_1,T})e^{-r(T-t_1)}]$$

– Approximate value of portfolio if $T - t_1$ is small, as is generally the case, then $e^{-r(T-t_1)} \approx 1$ and we have

$$N_A[S_{t_1} + h(F_{t_0,T} - F_{t_1,T})]$$

which is the expression we will use.

– "Per extended unit value"
  * Each unit of the asset is accompanied by h units of a futures contract
  * Call $S_{t_1} + h(F_{t_0,T} - F_{t_1,T})$ the value of an "extended unit" of the asset.
  * All the risk arises from this value, not from N_A, so we analyze the behavior of $S_{t_1} + h(F_{t_0,T} - F_{t_1,T})$.

3. Riskiness of the value
   - At t_0, when the futures contract is written, the values of S_{t_1} and $F_{t_1,T}$ are unknown random variables.
   - The change in the value of the extended unit is

$$\begin{aligned} & [S_{t_1} + h(F_{t_0,T} - F_{t_1,T})] - [S_{t_0} + h(F_{t_0,T} - F_{t_0,T})] \\ = \; & [S_{t_1} + h(F_{t_0,T} - F_{t_1,T})] - [S_{t_0} + 0] \\ = \; & (S_{t_1} - S_{t_0}) + h(F_{t_0,T} - F_{t_1,T}) \\ = \; & \Delta S - h\Delta F \end{aligned}$$

where $\Delta S \equiv S_{t_1} - S_{t_0}$ and $\Delta F \equiv F_{t_1,T} - F_{t_0,T}$.

– This change is a random variable with variance equal to

$$\begin{aligned} & Var\left[\Delta S - h\Delta F\right] \\ = \; & Var[\Delta S] + h^2 Var[\Delta F] - 2hCov[\Delta S, \Delta F] \\ = \; & \sigma_S^2 + h^2\sigma_F^2 - 2h\sigma_{SF} \\ = \; & \sigma_S^2 + h^2\sigma_F^2 - 2h\,\rho\,\sigma_S\,\sigma_F \end{aligned}$$

Note that σ_S, σ_F, ρ are functions of t_1.

– Notice that the same expression applies to a long hedge

∗ Change in value of payment to be made is

$$(S_{t_1} - S_{t_0}) + h \underbrace{(F_{t_0,T} - F_{t_1,T})}_{\text{loss on futures contract}}$$

∗ Same variance expression

– Choose h to minimize the variance

$$\begin{aligned}\frac{\partial Var}{\partial h} &= 2h\sigma_F^2 - 2\rho\sigma_S\sigma_F \\ &= 0 \quad \text{for a minimum}\end{aligned}$$

Which gives

$$\begin{aligned}h^\star &= \rho\frac{\sigma_S}{\sigma_F} \\ &= \text{regression coefficient from regressing } S \text{ on } F\end{aligned}$$

– Optimum number of contracts

Define:

* N_A = size of position being hedged (i.e., number of units of the asset being hedged).
* Q_F = size of one futures contract (i.e., number of units of asset in one futures contract).
* $N^\star$ = optimal number of futures contracts.

Value of $N^\star$ is

$$N^\star = \frac{h^\star N_A}{Q_F}$$