13 Implied Volatility

- Option prices are a function of: K, S_0, T, r, σ.
- All of these can be directly observed on the market except σ.
- We also observe the actual option prices on the market (c, C, p, P)
- If we assume that S_t follows a GBM, then

$$c = BS_c(S_0, K, T, r, \sigma)$$
$$p = BS_p(S_0, K, T, r, \sigma)$$

- Since we observe everything except σ we can solve for σ:

$$BS^{-1}(\{c \text{ or } p\}, S_0, K, T, r) = \sigma_{IV}$$

- The implied volatility, σ_{IV}, is the σ consistent with the observed $(\{c \text{ or } p\}, S_0, K, T, r)$ if Black-Scholes' setup is true.
- In practice, computing σ_{IV} is easy because $\frac{\partial BS}{\partial \sigma} > 0$.
- So, a first method to compute/estimate/calibrate σ is to look at the option price.

- A second method to compute/estimate/calibrate σ is to use historical data on the stock price S_t.

- If S_t follows a GBM, then

$$\ln S_T - \ln S_t \sim N\left(\left(\mu - \frac{\sigma^2}{2}\right)(T-t), \sigma^2(T-t)\right)$$

- If t is today and $t-1$ is yesterday, we have

$$\begin{aligned} \ln S_t - \ln S_{t-1} &\sim N\left(\left(\mu - \frac{\sigma^2}{2}\right)\frac{1}{252}, \sigma^2 \frac{1}{252}\right) \\ &\sim N(\bar{\mu}, \bar{\sigma}^2) \end{aligned}$$

- If we:
  - Get historical data on S_t.
  - Compute $y_t = \ln S_t - \ln S_{t-1}$ for $t = 1, 2, \ldots, N$.
  - Compute $\widehat{Var(y_t)} = \frac{1}{N}\sum_{t=1}^{N}(y_t - \bar{y})^2$ with $\bar{y} = \frac{1}{N}\sum_{t=1}^{N} y_t$.
  - We see that $\widehat{Var(y_t)} = \widehat{\bar{\sigma}}^2$
  - Since $\bar{\sigma}^2 = \frac{\sigma^2}{252}$ we take $\hat{\sigma}^2 = 252\,\widehat{\bar{\sigma}}^2$.
  - for example, if I take S_t to be the daily value of the S&P500 index from November 6, 2000 to November 4, 2005 I get $\hat{\sigma}^2 = 0.1878$.

- But many things are not right:
  - If I have 20 options traded on the same underlying asset and I compute the implied volatilities, I will get 20 different implied volatilities. I should get the “same” number all the time. What I tend to see is a volatility smile.
  - If I take historical volatility and I compute option prices using BS formula, then actual prices and BS prices are not the same (especially puts and calls deep in-the-money).

1. Do we conclude that BS is useless?
2. What is wrong in the BS setup (GBM)?
3. Why BS implied volatilities are (very) useful?

1. Is BS useless? No. It can be used as a filter to extract information about the market. The following will give you an idea why.

- Remember that we got the put-call parity without mentioning anything about evolution of S_t.

$$p + S_0 = c + Ke^{-rT}$$

- If we assume GBM for S_t, then we must have

$$p_{BS} + S_0 = c_{BS} + Ke^{-rT}$$

- By absence of arbitrage, prices on the market must also satisfy put-call parity:

$$p_{mkt} + S_0 = c_{mkt} + Ke^{-rT}$$

- subtract the two:

$$p_{mkt} + p_{BS} = c_{mkt} - c_{BS}$$

- When you see this equation you can think like this:
  - Take c_{mkt} and find σ_{IV}, the σ that for which $c_{mkt} = c_{BS}$
  - Once I have σ_{IV}, I can price other options. p_{BS} using σ_{IV} should give me p_{mkt}.
- Of course I can get p_{mkt} from c_{mkt} directly using put-call parity but the above reasoning gives us another approach.

2. What is wrong with BS setup (GBM)?

- The GBM says that $\ln S_t - \ln S_{t-1} \sim N(\mu, \sigma^2)$.
- some things that are different in real life:
  - σ^2. Volatility is not constant. Periods of high (low) volatility tend to be followed by periods of high (low) volatility.
  - Normal distribution. The Normal distribution has thin tails, they shrink at rate $\exp(-\frac{1}{2}x^2)$ since $f(x) = \frac{1}{\sqrt{2\pi}} exp\left(-\frac{1}{2}x^2\right)$.
    - Not enough mass in the extremes
    - does not allow for jumps
  - No leverage effect: For stocks, when the stock price goes down, the level of debt stays the same so the leverage increases. When stock price goes down the volatility of the stock price goes up to reflect that it is more risky (vice versa).

3. Why BS implied volatility can be (very) useful?

- Google “Practitioner Black-Scholes”
- σ_{IV} should be constant but it changes as we look at different maturity and strike prices (same underlying asset). Let’s try to capture this relationship.

- We do the following:
  1. Take a set of options on the same underlying stock. Compute their implied volatilities.
  2. Take these implied volatilities and run the following regression:

     $$\sigma_{IV} \;=\; \theta_0 + \theta_1 K + \theta_2 K^2 + \theta_3 T + \theta_4 T^2 + \theta_5 K \times T + e_{IV}$$

     and get estimates $(\hat{\theta}_0, \ldots, \hat{\theta}_5)$.
  3. Using these estimates, compute the predicted implied volatility for an option with a given K and T:

     $$\hat{\sigma}_{IV} \;=\; \hat{\theta}_0 + \hat{\theta}_1 K + \hat{\theta}_2 K^2 + \hat{\theta}_3 T + \hat{\theta}_4 T^2 + \hat{\theta}_5 K \times T$$

     and plug $\hat{\sigma}_{IV}$ in BS' formula.