8 Interest Rates

Bond: you pay now to receive an amount later (the face value) and possibly coupon payments.

8.1 Zero rates

- The n-year zero rate (short for zero-coupon rate) is the rate of interest earned on a zero-coupon investment that starts today and lasts n years.
- Interest rates are usually function of the maturity.

- Formula for T-period asset:

$$\begin{aligned}
P_0 e^{rT} &= P_T \\
e^{rT} &= \frac{P_T}{P_0} \\
rT &= \ln\left(\frac{P_T}{P_0}\right) \\
r &= \frac{1}{T}\ln\left(\frac{P_T}{P_0}\right) \\
&\approx \frac{1}{T}\frac{P_T - P_0}{P_0} \\
&= \text{average percentage change in price over } T \text{ periods}
\end{aligned}$$

8.2 Coupon rate

- The coupon rate is the stated (contracted) rate used to determine periodic payments of income during the life of the bond.
- Formula for bond price:

$$B = \frac{cP}{1+r_1} + \frac{cP}{(1+r_1)(1+r_2)} + \cdots + \frac{cP}{\prod_{i=1}^{T}(1+r_i)} + \frac{P}{\prod_{i=1}^{T}(1+r_i)}$$

where

  – B = current value of bond

  – c = coupon rate

  – P = principal (face value)

  – r_i = 1-period market interest rate in period i. These are **forward rates** (discussion later on in this section).

  – T = term to maturity

8.3 Bond yield

- The bond yield is the **constant** discount rate that equates the present value of the bond 's cash flow to the bond's market value.
- Formula:

$$B = \frac{cP}{1+r^\star} + \frac{cP}{(1+r^\star)^2} + \cdots + \frac{cP}{(1+r^\star)^T} + \frac{P}{(1+r^\star)^T}$$

where

$$B = \frac{cP}{1+r_1} + \frac{cP}{(1+r_1)(1+r_2)} + \cdots + \frac{cP}{\prod_{i=1}^{T}(1+r_i)} + \frac{P}{\prod_{i=1}^{T}(1+r_i)}$$

8.4 Par yield

- The par yield is a coupon rate that makes the bond's market value equal its face value.
- Formula:

$$\begin{aligned} P &= B^\star \\ &= \frac{c^\star P}{1+r_1} + \frac{c^\star P}{(1+r_1)(1+r_2)} + \cdots + \frac{c^\star P}{\prod_{i=1}^{T}(1+r_i)} + \frac{P}{\prod_{i=1}^{T}(1+r_i)} \end{aligned}$$

Canceling P we get

$$1 = \frac{c^\star}{1+r_1} + \frac{c^\star}{(1+r_1)(1+r_2)} + \cdots + \frac{c^\star}{\prod_{i=1}^{T}(1+r_i)} + \frac{1}{\prod_{i=1}^{T}(1+r_i)}$$

8.5 Computing treasury zero rates: the bootstrap method

- We want to know the zero rates implied by the rate of existing bonds.
- The problem is that existing bonds usually pay coupons.
- For example, suppose the current data are

bond principal	time to maturity	semi annual coupon	bond price
\$ 100	0.25	0	\$ 97.5
\$ 100	0.50	0	\$ 94.9
\$ 100	1.0	0	\$ 90.0
\$ 100	1.5	\$ 4	\$ 96.0
\$ 100	2.0	\$ 6	\$ 101.6

1. Start with the shortest existing bond and compute its zero rate:
   - The 3-month bond pays \$100 in 3 months on a \$97.50 investment.
   - Annual interest rate $r_{0.25}$ is determined from the formula:

$$\begin{aligned} 100 &= 97.5\left(1+\frac{r_{0.25}}{4}\right) \\ r_{0.25} &= (\frac{100}{97.5}-1)4 \\ &\approx 0.10256 \end{aligned}$$

   - Convert to continuous compounding

$$\begin{aligned} e^{r^{\star}_{0.25}} &= \left(1+\frac{0.10256}{4}\right)^{4} \\ r^{\star}_{0.25} &= 4\ln\left(1+\frac{0.10256}{4}\right) \\ &= 0.10127 \end{aligned}$$

2. Go to the next bond

   - The 6-month bond pays \$100 in 6 months on a \$94.90 investment
   - Annual interest rate $r_{0.5}$ is determined from the formula:

$$\begin{aligned} 100 &= 94.9\left(1+\frac{r_{0.5}}{2}\right) \\ r_{0.5} &= (\frac{100}{94.9}-1)2 \\ &\approx 0.10748 \end{aligned}$$

   - Convert to continuous compounding

$$\begin{aligned} e^{r^{\star}_{0.5}} &= \left(1+\frac{0.10748}{2}\right)^2 \\ r^{\star}_{0.5} &= 2\ln\left(1+\frac{0.10748}{2}\right) \\ &= 0.10469 \end{aligned}$$

3. Go to the next bond

- The 1-year bond pays \$100 in 6 months on a \$90.00 investment
- Annual interest rate $r_{1.0}$ is determined from the formula:

$$\begin{aligned} 100 &= 90.0(1 + r_{1.0}) \\ r_{1.0} &= \frac{100}{90.0} - 1 \\ &\approx 0.11111 \end{aligned}$$

- Convert to continuous compounding

$$\begin{aligned} e^{r^\star_{1.0}} &= (1 + 0.11111) \\ r^\star_{1.0} &= \ln(1.11111) \\ &= 0.10536 \end{aligned}$$

4. Go to the next bond

   - The 1.5-year bond pays \$4 after 6 months, another \$4 after 1 year, and \$104 after 18 months, all on an investment of \$96.
   - For the bond to be priced correctly, the bond's price must equal the present value of its payments:

$$\begin{aligned} 96 &= 4\,e^{-0.5r^\star_{0.5}} + 4\,e^{-1.0r^\star_{1.0}} + 104\,e^{-1.5r^\star_{1.5}} \\ &= 4\,e^{-0.5(0.10469)} + 4\,e^{-1.0(0.10536)} + 104\,e^{-1.5r^\star_{1.5}} \end{aligned}$$

   The only unknown is $r^\star_{1.5}$, so we solve for it to obtain

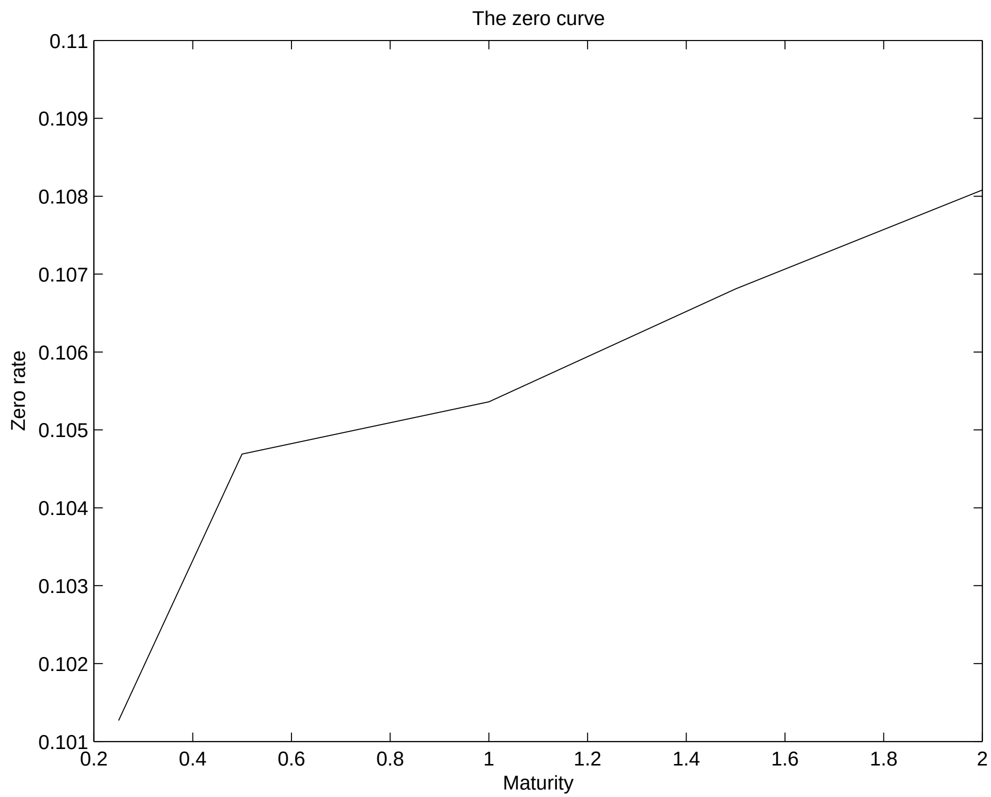
$$r^\star_{1.5} = 0.10681$$

5. Go to the last bond and follow the same procedure

   - We get:

$$
\begin{aligned}
101.6 &= 6\,e^{-0.5(0.10469)} + 6\,e^{-1.0(0.10536)} + 6\,e^{-1.5(0.10681)} + 106\,e^{-2.0 r^\star_{2.0}} \\
r^\star_{2.0} &= 0.10808
\end{aligned}
$$

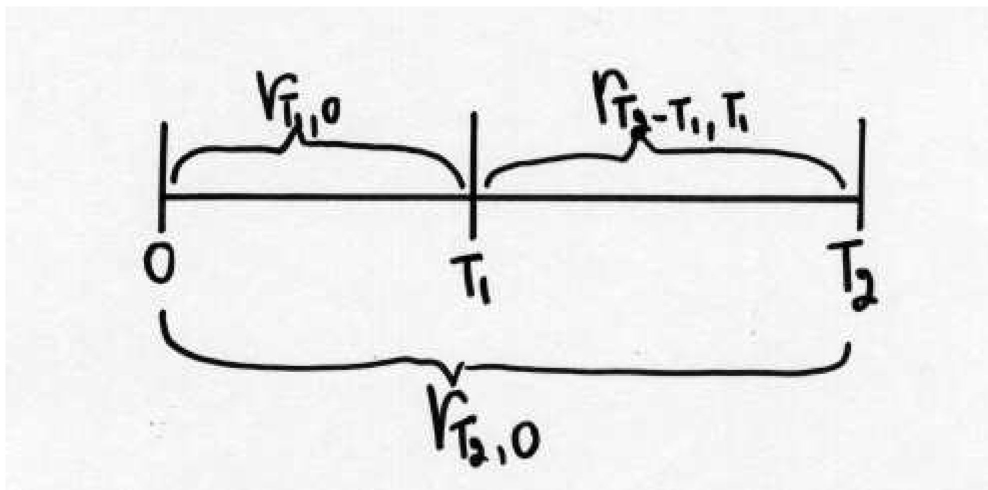
6. The zero curve



8.6 Forward rates

- **Definition**: the rates of interest for future time periods that are implied by today's interest rates.
- **Calculation**
  1. Define r_{ij} the interest rate prevailing in period j for the period i:
     - $r_{1,1}$ = 1-period rate prevailing in period 1
     - $r_{2,1}$ = 2-period rate (i.e., rate on 2-period bond in period 1)
     - $r_{1,2}$ = 1-period rate in period 2

2. Relation among rates



$$
\begin{aligned}
& e^{t_2 r_{t_2,0}} = e^{t_1 r_{t_1,0}} e^{(t_2-t_1) r_{t_2-t_1,t_1}} \\
\Leftrightarrow \quad & e^{t_2 r_{t_2,0}} e^{-t_1 r_{t_1,0}} = e^{(t_2-t_1) r_{t_2-t_1,t_1}} \\
\Leftrightarrow \quad & e^{t_2 r_{t_2,0} - t_1 r_{t_1,0}} = e^{(t_2-t_1) r_{t_2-t_1,t_1}} \\
\Leftrightarrow \quad & \frac{t_2 r_{t_2,0} - t_1 r_{t_1,0}}{t_2 - t_1} = r_{t_2-t_1,t_1} \\
\Leftrightarrow \quad & r_{t_2-t_1,t_1} = r_{t_2,0} + \left(r_{t_2,0} - r_{t_1,0}\right) \frac{t_1}{t_2 - t_1} \\
\Rightarrow \quad & \begin{cases} r_{t_2-t_1,t_1} > r_{t_2,0} & \text{if} \quad r_{t_2,0} > r_{t_1,0} \\ r_{t_2-t_1,t_1} < r_{t_2,0} & \text{if} \quad r_{t_2,0} < r_{t_1,0} \end{cases}
\end{aligned}
$$

Conclusion 1: the long rate ($r_{t_2,0}$) lies between the current ($r_{t_1,0}$) and forward short rates ($r_{t_2-t_1,t_1}$).

We can also write:

$$r_{t_2,0} = \frac{t_1}{t_2} r_{t_1,0} + \frac{t_2 - t_1}{t_2} r_{t_2-t_1,t_1}$$

Conclusion 2: the long rate is a weighted average of the short rates.

8.7 Theories of the term structure

1. Segmentation theory:
   - No link between short-, medium- and long-term interest rates
   - There is a supply and demand of short-term bonds that determine the short rate. Same thing for medium and long term.
2. Expectation theory:

$$r_{t_2,0} = \frac{t_1}{t_2} r_{t_1,0} + \frac{t_2 - t_1}{t_2} E[r_{t_2-t_1,t_1}]$$

3. Liquidity preference theory:
   (a) Asymmetry of bankruptcy and default
      - Borrower may default in bad times
      - No offsetting gain to lender in good times
   (b) Longer-term loans have more chance of default
   (c) This leads to asymmetric preferences
      - Borrowers like to borrow long
      - Lenders like to lend short
   (d) Inequality between demand and supply if interest rates are the same for short and long loans
      - Excess supply in short-term market
      - Excess demand in long-term market
   (e) This forces the short rate to be smaller than the long rate.

8.8 Duration

1. **Definition 1**: a measure of how long on average the bond holder must wait for payment.
2. **Definition 2**: a measure of the asset's price sensitivity to a change in the yield (better definition).

3. **Calculation**

- Zero coupon bond:

$$\begin{aligned} D &\equiv \text{duration} \\ &= \text{maturity of the bond} \end{aligned}$$

- Coupon bond – more complex:

(a) Define:

B = bond price

y = yield (continuously compounded)

c_i = cash flow at time t_i

(b) Bond price:

$$B = \sum_{i=1}^{n} c_i e^{-y t_i}$$

(c) Duration formula:

$$
\begin{aligned}
D &= \frac{\sum_{i=1}^{n} t_i c_i e^{-y t_i}}{B} \\
&= \frac{\text{time-weighted present value}}{\text{present value}} \\
&= \sum_{i=1}^{n} t_i \left(\frac{c_i e^{-y t_i}}{B} \right) \\
&= \text{weighted average of times to payments}
\end{aligned}
$$

since the term between parentheses in the next to last equation is the fraction of total present value due to ith payment.

4. **Duration and yield sensitivity of bond prices**

$$\begin{aligned}\frac{dB}{dy} &= -\sum_{i=1}^{n} c_i t_i e^{-y t_i} \\ &= -B\,D \quad \text{(look at equation above)} \\ \Rightarrow \frac{dB}{B} &= -D dy\end{aligned}$$

Which means the percentage change in the bond price is the change in yield multiplied by the bond duration. [The accuracy of this relation is guaranteed only for small values of dy.]

This gives a quick way to assess sensitivity of bond value to change in yield.

5. **Duration and bond portfolios**

   The duration of a bond **portfolio** can be defined as the weighted average of the individual bonds' durations, with weights proportional to the bond price:

$$
\begin{aligned}
D &= \sum_{i=1}^{N} \left(\frac{B_i}{\sum_{j=1}^{N} B_j} \right) D_i \\
&= \frac{\sum_{i=1}^{N} B_i D_i}{\sum_{j=1}^{N} B_j} \\
&= \frac{\sum_{i=1}^{N} B_i D_i}{B} \qquad \text{for } B = \sum_{j=1}^{N} B_j.
\end{aligned}
$$

Then,

$$\frac{dB}{B} = -Ddy$$

if we assume that all bond yields y_i change by the same amount (i.e., parallel shift in the yield curve).

6. **Duration-based hedging**

   (a) Suppose an interest-rate dependent asset (such as a bond portfolio) is being hedged using an interest rate futures contract.

   (b) Define:

F_c = contract price for the interest rate futures contract (one contract delivers \$100,000 face value of the bond in question)

D_F = duration of the asset underlying the futures contract at the maturity of the futures contract

P = forward value of the portfolio being hedged at the maturity of the hedge (in practice, usually taken to be the value of the portfolio today).

D_P = duration of the portfolio at the maturity of the hedge.

(c) If dy is the same for all maturities, then

$$\begin{aligned} dP &\approx -PD_P dy \\ dF_C &\approx -F_C D_F dy \end{aligned}$$

Then, the number of contracts required to hedge against an uncertain dy is

$$N^\star = \frac{PD_P}{F_C D_F}$$

because we want to hold $N^\star$ futures contracts so that $N^\star dF_c = dP$.

$N^\star$ is the **duration-based hedge ratio** (or the price sensitivity hedge ratio).

8.9 Convexity

- This measures how fast $\frac{dB}{dy}$ is changing with the size of dy.

$$C \equiv \frac{1}{B}\frac{d^2B}{dy^2} = \frac{\sum_{i=1}^{n} c_i t_i^2 e^{-yt_i}}{B}$$

- Use 2nd order Taylor series to write

$$\begin{aligned} dB &= -\sum_{i=1}^{n} c_i t_i e^{-yt_i} dy + \frac{1}{2}\sum_{i=1}^{n} c_i t_i^2 e^{-yt_i} (dy)^2 \\ &= -BDdy + \frac{1}{2}BC(dy)^2 \\ \Rightarrow \frac{dB}{B} &= -Ddy + \frac{1}{2}C(dy)^2 \end{aligned}$$

One can use convexity to improve a hedge and immunize against fairly large parallel shifts in the zero curve.