12 Black-Scholes

12.1 The fundamental PDEs

1. **Stock process** (Geometric Brownian Motion)

$$dS = \mu S\, dt + \sigma S\, dz$$

2. **Option process**

$$dF = \left(\mu S \frac{\partial F}{\partial S} + \frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} \right) dt + \sigma S \frac{\partial F}{\partial S} dz$$

3. **Riskless portfolio**

- The foregoing dF equation is difficult because of the random term.
- Convenient to eliminate that term by constructing a riskless portfolio:
  - Go short one derivative and go long N shares of S
  - Portfolio value is

$$V = -F + NS$$

Its change is

$$\begin{aligned} dV &= -dF + NdS \\ &= -\left(\mu S \frac{\partial F}{\partial S} + \frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2}\right) dt - \sigma S \frac{\partial F}{\partial S} dz \\ &\quad + N\mu S\, dt + N\sigma S dz \end{aligned}$$

  - To make this portfolio riskless, choose $N = \frac{\partial F}{\partial S}$

$$\Rightarrow dV = -\left(\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2}\right) dt$$

4. **Equality with riskless asset return**

- The last equation is the return from holding a risk-free portfolio.
- Absence of arbitrage guarantee that this return equals the return from putting the value of the portfolio into the risk-free asset:

$$dV = rV\,dt$$

- We therefore must have

$$-\left(\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2}\right) dt \quad = \quad r\left(-F + \frac{\partial F}{\partial S} S\right) dt$$

- Which can be written as

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} + rS\frac{\partial F}{\partial S} - rF + \frac{\partial F}{\partial t} = 0$$

- This is the fundamental equation for all derivative pricing, known as the Black-Scholes (-Merton) equation. Solving this gives F.

- The Black-Scholes equation applies to any derivative asset. What distinguishes the different assets are the boundary conditions:
  - At $S = 0$, we have $dS = 0$, because $dS = \mu S\,dt + \sigma S\,dz$.
  - The other boundary condition occurs at the expiration (or exercise point in the case of an American option) and is different for each derivative asset.
  - For example, a call option's condition is

$$F(S, T) = \max(S_T - K, 0)$$

12.2 Solving the equation

1. **Two approaches:**
   - PDE with boundary conditions
   - Risk-neutral valuation (martingale theory). This approach is more convenient for this problem.
2. **Risk-neutral valuation**
   (a) Applies here because
   - There is a finite stopping time (the expiration date)
   - No element of investor risk preference enters the Black-Scholes equation.
     – Note in particular that the expected return μ on the stock is absent from the equation. The value of μ does depend on risk preferences.

(b) Useful implication

- If risk preferences do not enter the equation, they do not affect the solution.
- $\Rightarrow$ Any set of risk preferences can be used when evaluating F.
- $\Rightarrow$ We can use risk-neutral preferences
- $\Rightarrow$ Discounting can be done with the risk-free rate r.

(c) Simple procedure

- Assume the expected return from the underlying asset is the risk-free rate (i.e, replace μ by r).
- Calculate the expected payoff from the option at maturity.
- Discount the expected payoff at the risk-free rate.

(d) Note:

- The assumption of risk-neutral preferences is a convenience. The resulting solution is valid for any preferences, including those in the real world.
- When we move from risk-neutral to risk-averse preferences, two things happen:
  - The expected growth rate of the stock changes
  - The discount rate changes

  It turns out these two effects always exactly offset each other.

(e) Simple example of how to use risk-neutral valuation to get a derivative asset price:

- Take a long forward contract with maturity date T and delivery price K.
- Value of the contract at maturity is $S_T - K$
- Therefore, the value at time 0 is

$$\begin{aligned} F &= e^{-rT} E^\star \left[S_T - K \right] \\ &= e^{-rT} E^\star \left[S_T \right] - K e^{-rT} \end{aligned}$$

- Under risk-neutrality, the growth rate of the stock μ equals r, so we can write

$$E^\star[S_T] = S_0 e^{rT}$$

- Substitute this result into the preceding expression for F to get

$$F = S_0 - K e^{-rT}$$

which agrees with what we had derived earlier in the semester.

3. **Preliminary lemma**

Lemma: If $\ln V \sim N(m, S^2)$, then

$$E\left[\max(V-K,0)\right] = E[V]N(d_1) - KN(d_2)$$

where

$$d_1 \equiv \frac{\ln\left(\frac{E[V]}{K}\right) + \frac{S^2}{2}}{S}$$
$$d_2 \equiv \frac{\ln\left(\frac{E[V]}{K}\right) - \frac{S^2}{2}}{S}$$

and $N(x)$ is the cumulative distribution function of a standardized normal distribution evaluated at x:

$$N(x) = \int_{-\infty}^{x} n(u)du \qquad \text{and} \qquad n(u) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}u^2}$$

Proof: Let $g(V)$ be the probability density of V. Then

$$E\left[\max(V-K,0)\right] = \int_{K}^{\infty} (V-K)g(V)dV$$

By assumption, $\ln V \sim N(m, S^2)$ and by the properties of the log-normal distribution, the mean of $\ln V$ is m,

$$m = \ln(E[V]) - S^2/2$$

with the mean of V being $e^{m+S^2/2}$.

Define the standardized normal variable

$$Q \equiv \frac{\ln V - m}{S}$$

with density

$$h(Q) = \frac{1}{\sqrt{2\pi}} e^{-Q^2/2}$$

We can then use a change of variables to write

$$\begin{aligned} E\left[\max(V-K,0)\right] &= \int_{\frac{\ln K-m}{S}}^{\infty}\left(e^{QS+m}-K\right)h(Q)dQ \\ &= \int_{\frac{\ln K-m}{S}}^{\infty} e^{QS+m}h(Q)dQ - K\int_{\frac{\ln K-m}{S}}^{\infty} h(Q)dQ \end{aligned}$$

But

$$\begin{aligned} e^{QS+m}h(Q) &= \frac{1}{\sqrt{2\pi}}e^{\frac{1}{2}(-Q^2+2QS+2m)} \\ &= \frac{1}{\sqrt{2\pi}}e^{\frac{1}{2}(-(Q-S)^2+2m+S^2)} \\ &= \frac{e^{m+S^2/2}}{\sqrt{2\pi}}e^{\frac{-1}{2}(Q-S)^2} \\ &= e^{m+\frac{S^2}{2}}h(Q-S) \end{aligned}$$

Therefore

$$
\begin{aligned}
& E\left[\max(V-K,0)\right] \\
= \; & e^{m+\frac{S^2}{2}} \int_{\frac{\ln K-m}{S}}^{\infty} h(Q-S)dQ - K\int_{\frac{\ln K-m}{S}}^{\infty} h(Q)dQ \\
= \; & e^{m+\frac{S^2}{2}} \left[1-N\left(\frac{\ln K-m}{S}-S\right)\right] - K\left[1-N\left(\frac{\ln K-m}{S}\right)\right] \\
= \; & e^{m+\frac{S^2}{2}} \left[N\left(\frac{-\ln K+m}{S}+S\right)\right] - K\left[N\left(\frac{-\ln K+m}{S}\right)\right] \\
& \qquad \text{because } N \text{ is symmetric about zero} \\
= \; & e^{m+\frac{S^2}{2}} \left[N\left(\frac{\ln(\frac{E[V]}{K})+\frac{S^2}{2}}{S}\right)\right] - K\left[N\left(\frac{\ln(\frac{E[V]}{K})-\frac{S^2}{2}}{S}\right)\right] \\
& \qquad \text{by substituting for } m \\
= \; & e^{m+\frac{S^2}{2}} N(d_1) - KN(d_2) \\
= \; & E[V]N(d_1) - KN(d_2) \\
& \qquad \text{by substituting for } m \text{ again}
\end{aligned}
$$

4. **Main result**

Theorem: A European call option on a non-dividend stock with maturity date T and strike price K has a price of

$$c = S_0 N(d_1) - K e^{-rT} N(d_2)$$

where

$$\begin{aligned} d_1 &= \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \\ d_2 &= \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \end{aligned}$$

Proof: The value of the call is

$$c = e^{-rT} E^{\star} \left[\max(S_T - K, 0)\right]$$

Because the stochastic process for S is

$$dS = \mu S \, dt + \sigma S \, dz$$

we know S is log-normal. Also, from earlier results, we know that

$$\begin{aligned} E^{\star}[S_T] &= S_0 e^{rT} \\ Var\left[\ln S_T\right] &= \sigma^2 T \end{aligned}$$

So using the previous lemma,

$$\begin{aligned} c &= e^{-rT} E^{\star} \left[\max(S_T - K, 0)\right] \\ &= e^{-rT} \left[S_0 e^{rT} N(d_1) - K N(d_2)\right] \\ &= S_0 N(d_1) - K e^{-rT} N(d_2) \end{aligned}$$

where

$$
\begin{aligned}
d_1 &= \frac{\ln\left(\frac{E^\star[S_T]}{K}\right) + \frac{\sigma^2+T}{2}}{\sigma\sqrt{T}} \\
&= \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \\
d_2 &= \frac{\ln\left(\frac{E^\star[S_T]}{K}\right) - \frac{\sigma^2+T}{2}}{\sigma\sqrt{T}} \\
&= \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}
\end{aligned}
$$

5. **Put option**

- Similar reasoning established the price for a European put option:

$$p = Ke^{-rT}N(-d_2) - S_0N(-d_1)$$

where d_1 and d_2 are the same as for the call option

- An alternative proof is to use the put-call parity:

$$\begin{aligned} p + S_0 &= c + Ke^{-rT} \\ \Rightarrow p &= S_0N(d_1) - Ke^{-rT}N(d_2) + Ke^{-rT} - S_0 \\ &= -S_0\left[1 - N(d_1)\right] + Ke^{-rT}\left[1 - N(d_2)\right] \\ &= -S_0N(-d_1) + Ke^{-rT}N(-d_2) \end{aligned}$$

since the normal distribution is symmetric