2 Present Value

- If you have to decide between receiving 100$ now or 100$ one year from now, then you would rather have your money now.
- If you have to decide between paying 100$ now or 100$ one year from now, then you would rather pay one year from now.
- This section is about computing the value today of something in the future.

2.1 Definition

Present value: the value today of an amount to be paid at a specific date in the future.

2.2 Simple case

- One payment of face amount (principal) X, one period in the future.
  - First, suppose you had V now and could save it at interest rate R. In one period, you would have $X = V(1+R)$.
  - Rearrange terms to get $V = \frac{X}{1+R}$.
  - V is the **present value** of X.
- Two periods in the future:

$$X = V(1+R_1)(1+R_2) = V \prod_{i=1}^{2}(1+R_i)$$

from which we get:

$$V = X\left[\prod_{i=1}^{2}(1+R_i)\right]^{-1} = X\left[\prod_{i=1}^{2}(1+R_i)^{-1}\right]$$

- Many periods:

$$V = X\left[\prod_{i=1}^{I}(1+R_i)^{-1}\right]$$

- Special case when $R_1 = R_2 = \cdots = R_I = R$:

$$V = \frac{X}{(1+R)^I}$$

2.3 More complicated cases – interim payments

$$
\begin{aligned}
V &= C_0 + \frac{C_1}{1+R_1} + \frac{C_2}{(1+R_1)(1+R_2)} + \cdots \\
&\quad + \frac{C_I}{(1+R_1)(1+R_2)\cdots(1+R_I)} \\
&\quad + \frac{X}{(1+R_1)(1+R_2)\cdots(1+R_I)} \\
&= \sum_{i=0}^{I}\left[C_i \prod_{j=0}^{i}(1+R_j)^{-1}\right] + X\prod_{i=0}^{I}(1+R_j)^{-1}
\end{aligned}
$$

provided that we define $R_0 \equiv 0$.

2.4 Continuous time

- Compounding:
  - $1+R \rightarrow (1+\frac{R}{2})(1+\frac{R}{2}) = (1+\frac{R}{2})^2$
  - $1+R \rightarrow (1+\frac{R}{n})^n$
  - $\lim_{n\rightarrow\infty}(1+\frac{R}{n})^n \equiv e^R$
  - $(1+R)^t \rightarrow [(1+\frac{R}{n})^n]^t = (1+\frac{R}{n})^{nt}$
  - $\lim_{n\rightarrow\infty}(1+\frac{R}{n})^{nt} = e^{Rt}$
- Present value of X to be paid at time T:

$$V = Xe^{-RT}$$

- Continuous-time vs. discrete-time interest rates:
  - Once per annum: $1+R = e^{\tilde{R}}$
  - n times per annum: $(1+\frac{R}{n})^n = e^{\tilde{R}}$

- Evaluating a continuous, constant, cash flow, R constant:

$$\begin{aligned} V &= \int_0^T X e^{-Rt} dt \\ &= X \int_0^T e^{-Rt} dt \\ &= X \left[-\frac{1}{R} e^{-Rt} \right]_0^T \\ &= X \left[-\frac{1}{R} e^{-RT} + \frac{1}{R} \right] \\ &= \frac{X}{R} (1 - e^{-RT}) \end{aligned}$$

Note that $V \rightarrow \frac{X}{R}$ as $T \rightarrow \infty$: perpetuity.

- Time-varying X:

$$V = \int_0^T X_t e^{-Rt} dt.$$

Can't say more until the function X_t is specified. For example, if $X_t = Ae^{Bt}$, then

$$\begin{aligned} V &= \int_0^T Ae^{Bt} e^{-Rt} dt \\ &= \int_o^T Ae^{-(R-B)t} dt \\ &= \frac{A}{R-B}(1 - e^{-(R-B)T}) \\ &\xrightarrow{T\to\infty} \begin{cases} \frac{A}{R-B} & \text{if} \quad R > B \\ \infty & \text{if} \quad R \leq B \end{cases} \end{aligned}$$

- Time-varying R, constant X:

$$V = \int_0^T X e^{-\int_0^t R_s ds} dt$$

(explanations follows).

- General case:

$$V = \int_0^T X_t e^{-\int_0^t R_s ds} dt$$

(explanation follows).

2.5 Deriving (and understanding) continuous discounting

2.5.1 Integrating factor

- We know how to compute derivatives. For example,

$$\frac{d\exp(x^7)}{dx} = 7x^6\exp(x^7)$$

- We then know that, up to an integrating constant,

$$\int 7x^6\exp(x^7)dx = \exp(x^7)$$

- The integral $\int\exp(x^7)dx$ might be hard to compute, but it would be much easier if we could somehow multiply $\exp(x^7)$ by $7x^6$.
- In this example, $7x^6$ is the integrating factor. More on this in a few slides.

2.5.2 Continuous discounting

- Earnings on an initial stock of wealth

$$\frac{dV_t}{dt} = RV_t$$

i.e., continuous compounding at rate R. To help you see this:

$$R = \frac{dV_t/dt}{V_t} \approx \frac{V_{t+\epsilon} - V_t}{V_t}$$

What is V at time T? To find out we solve the differential equation:

$$\frac{dV_t}{dt} - RV_t = 0.$$

The integrating factor is e^{-Rt} (how do we know that that's the integrating factor? We just know, or we wait a few more slides):

$$\begin{aligned} & \frac{dV_t}{dt}e^{-Rt} - V_t R e^{-Rt} = 0 \\ \Leftrightarrow \quad & \frac{d}{dt}(V_t e^{-Rt}) = 0. \end{aligned}$$

So we get

$$\begin{aligned} 0 &= \int_0^T \left[\frac{d}{dt}(V_t e^{-Rt})\right] dt \\ 0 &= [V_t e^{-Rt}]_0^T \\ 0 &= V_T e^{-RT} - V_0 \\ V_T &= V_0 e^{RT} \end{aligned}$$

- Now suppose there also is a time-varying income

$$\frac{dV_t}{dt} = RV_t + X_t$$

or

$$\frac{dV_t}{dt} - RV_t = X_t$$

which is just a forced (or non-homogeneous) linear ODE with the same integrating factor:

$$\begin{aligned} \frac{dV_t}{dt} e^{-Rt} - V_t R e^{-Rt} &= X_t e^{-Rt} \\ \frac{d}{dt}\left(V_t e^{-Rt}\right) &= X_t e^{-Rt}. \end{aligned}$$

So that we have

$$
\begin{aligned}
& \int_0^T \left[\frac{d}{dt}\left(V_t e^{-Rt}\right)\right] dt = \int_0^T X_t e^{-Rt} dt \\
\Rightarrow \quad & V_T e^{-RT} - V_0 = \int_0^T X_t e^{-Rt} dt \\
\Rightarrow \quad & V_T = V_0 e^{RT} + e^{RT} \int_0^T X_t e^{-Rt} dt.
\end{aligned}
$$

Note that if $V_0 = 0$, we have

$$\begin{aligned} V_T &= e^{RT} \int_0^T X_t e^{-Rt} dt \\ &= \int_0^T X_t e^{-R(t-T)} dt \end{aligned}$$

which is the future (time T) value of the cash flow X_t.

Equivalently,

$$V_T e^{-RT} = \int_0^T X_t e^{-Rt} dt$$

is the present value of the cash flow X_t.

- Finally, suppose R is time varying:

$$\begin{aligned} \frac{dV_t}{dt} &= R_t V_t + X_t \\ \Rightarrow \qquad \frac{dV_t}{dt} - R_t V_t &= X_t. \end{aligned}$$

The integrating factor is $e^{-\int_0^t R_s ds}$ so that we can write

$$\frac{dV_t}{dt} e^{-\int_0^t R_s ds} - V_t R_t e^{-\int_0^t R_s ds} = X_t e^{-\int_0^t R_s ds}$$

or

$$\frac{d}{dt}\left(V_t e^{-\int_0^t R_s ds}\right) = X_t e^{-\int_0^t R_s ds}.$$

So that

$$\begin{aligned}
\int_0^T \left[\frac{d}{dt}\left(V_t e^{-\int_0^t R_s ds}\right)\right] dt &= \int_0^T X_t e^{-\int_0^t R_s ds} dt \\
\left[V_t e^{-\int_0^t R_s ds}\right]_0^T &= \int_0^T X_t e^{-\int_0^t R_s ds} dt \\
V_T e^{-\int_0^T R_s ds} - V_0 &= \int_0^T X_t e^{-\int_0^t R_s ds} dt.
\end{aligned}$$

The RHS is the present value of $\{X_t\}$ when R varies over time.

2.6 Ordinary Differential Equation (ODE) and finding the integrating factor

- How do we know what the integrating factor is?
- Once you're familiar with the topic, you tend to "just know" what it is.
- But it does not mean that there are no formal ways to find it.
- The remaining slides give a general solution for the integrating factor for a specific class of ODE.
- Don't get lost in the details. This course is not about solving ODEs

- Consider a first-order first-degree ODE, which has the form

$$\frac{dy}{dx} = F(x, y)$$

which can be written as

$$M(x, y)dx + N(x, y)dy = 0 \;\Leftrightarrow\; \frac{dy}{dx} = -\frac{M(x, y)}{N(x, y)}$$

- If this equation has a unique solution, it can be written as

$$U(x, y) = c.$$

This is a general solution (allow implicit functions).

- If we take the differential on both sides

$$dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy = 0$$

- Which implies

$$\frac{dy}{dx} = -\frac{\partial U/\partial x}{\partial U/\partial y}$$

- So that

$$\frac{\partial U/\partial x}{\partial U/\partial y} = \frac{M}{N}$$

- Or

$$\frac{\partial U/\partial x}{M} = \frac{\partial U/\partial y}{N}$$

- Denote these ratios by μ,

$$\frac{\partial U}{\partial x} = \mu M \quad \text{and} \quad \frac{\partial U}{\partial y} = \mu N$$

- Next, substitute back into the equation for dU:

$$
\begin{aligned}
dU = \mu M dx + \mu N dy &= 0 \\
\mu \left(M dx + N dy \right) &= 0 \\
\frac{\partial U / \partial x}{M} M dx + \frac{\partial U / \partial y}{N} N dy &= 0 \\
dU = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy &= 0
\end{aligned}
$$

Conclusion 1: multiplying the differential equation $M(x,y)dx + N(x,y)dy = 0$ by μ give us an exact differential equation.

We call μ an integrating factor.

Before being able to say what μ is, we need an additional result:

- If the differential equation $M dx + N dy = 0$ is exact, then by definition, there is a function $U(x, y)$ such that

$$M dx + N dy = dU$$

- But we also have that

$$dU = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy$$

- Therefore, $\partial U / \partial x = M$ and $\partial U / \partial y = N$.

- For a sufficiently smooth function U,

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial^2 U}{\partial y \partial x} \\ &= \frac{\partial^2 U}{\partial x \partial y} \\ &= \frac{\partial N}{\partial x}\end{aligned}$$

- Thus, $\partial M/\partial y = \partial N/\partial x$ if the differential equation is exact.

We are ready to find μ, i.e. the integrating factor

- If the differential equation becomes exact after being multiplied by μ, i.e. $\mu M dx + \mu N dy$, then we have

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$$

- Next, suppose that μ is a function of x only (with a symmetric result if it is a function of y alone). In this case

$$\begin{aligned} \mu\frac{\partial M}{\partial y} &= \mu\frac{\partial N}{\partial x} + N\frac{d\mu}{dx} \\ \Leftrightarrow \quad \frac{d\mu}{\mu} &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) dx \\ &= f(x)dx \quad \text{by hypothesis} \\ \Rightarrow \mu &= e^{\int f(x)dx} \end{aligned}$$

Conclusion 2: If $M(x,y)dx + N(x,y)dy = 0$ and if

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x),$$

then $e^{\int f(x)dx}$ is the integrating function.

For the first example, we can find the integrating factor using these results.

- Remember we had

$$\frac{dV_t}{dt} = RV_t$$

- Which can be written

$$-RV_t dt + dV_t = 0$$

- We see that, remembering that we want to get V_t (take $y = V_t$)

$$x = t, \quad y = V_t, \quad M(x,y) = -RV_t, \quad N(x,y) = 1$$

- So we get

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{1}(-R - 0) = -R$$

- Hence, $\mu = e^{\int -Rdt} = e^{-Rt}$.

If you want more, have a look at the fine notes on ordinary differential and difference equations written by Professor John Seater. You'll find them on Professor Seater's website:

http://www4.ncsu.edu/∼jjseater

Or on the website of this course.