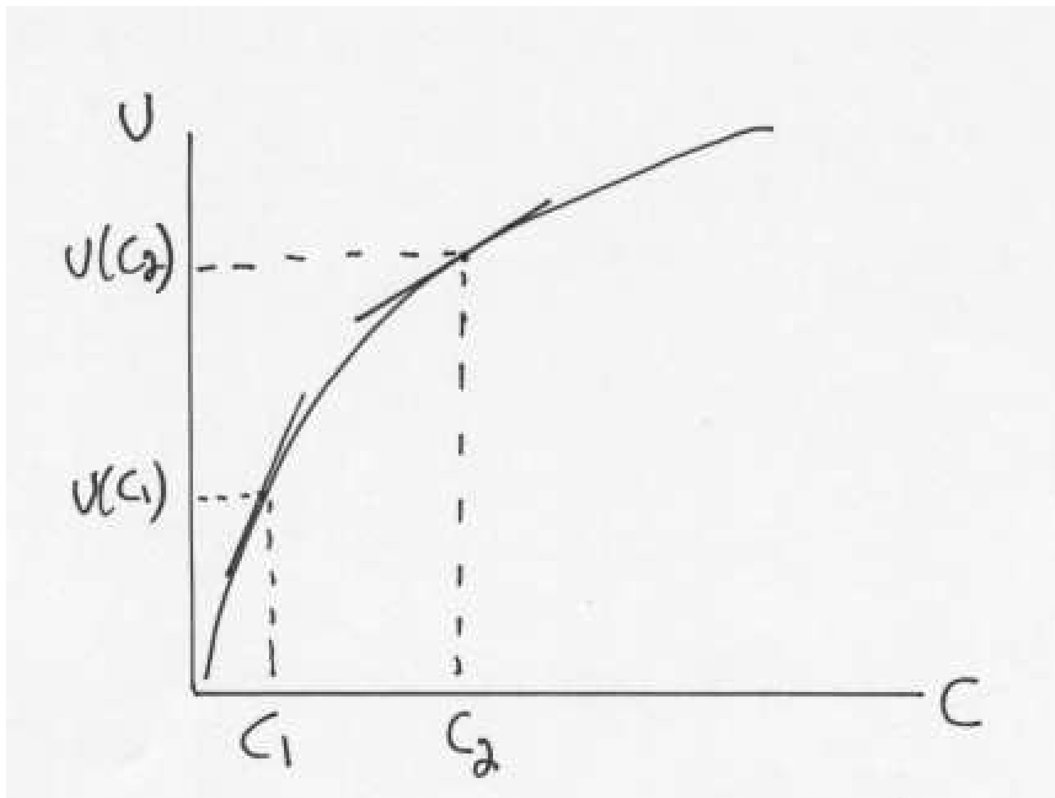
3 Risk Aversion

3.1 Utility

1. **Simple case**
   - Utility (happiness) depends on consumption
   - More consumption always raises utility
     - OK for total consumption
     - There are exceptions for particular goods (example: sun bathing)
   - The increase in utility brought about by a given increase in consumption is smaller the more consumption you already have.

2. **Mathematical representation**
   - Utility function: $U = U(C)$
   - Marginal utility: $\frac{dU}{dC} = U'(C) > 0$
   - Diminishing marginal utility: $\frac{d^2U}{dC^2} < 0$
   - These properties imply that the utility is a concave function



3.2 Risk aversion

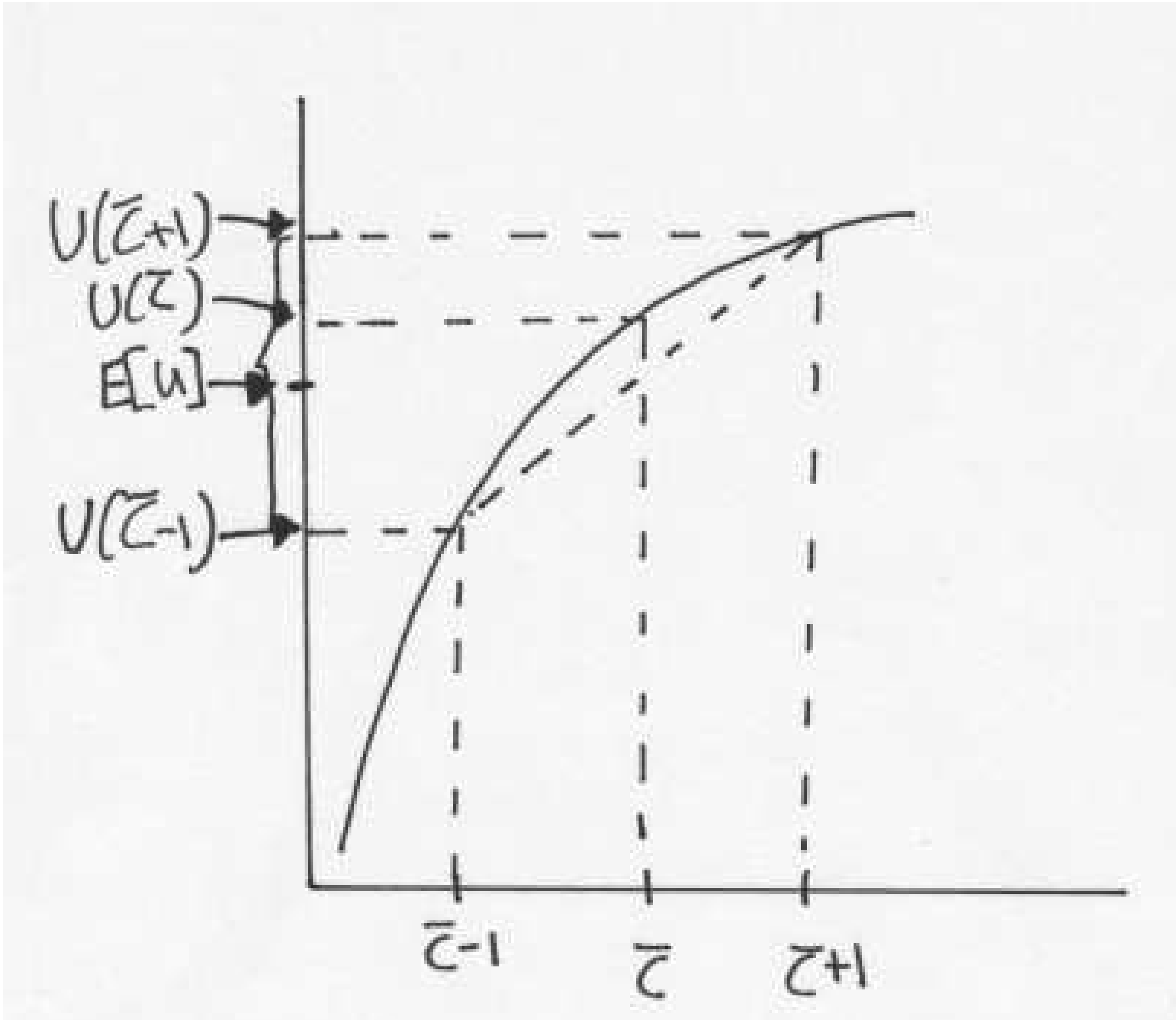
3.2.1 Basic intuition

- Imagine a simple lottery where you have a probability p of winning $\bar{C}-1$ and probability $q=1-p$ of winning $\bar{C}+1$. Suppose $p=0.5$
- The expected gain from the lottery is

$$\begin{aligned} E[C] &= 0.5(\bar{C}-1)+0.5(\bar{C}+1) \\ &= \bar{C} \end{aligned}$$

- The expected utility from the lottery is

$$\begin{aligned} E[U] &= 0.5U(\bar{C}-1)+0.5U(\bar{C}+1) \\ &< U(E[C]) \qquad \text{(by Jensen's inequality)} \\ &= U(\bar{C}) \end{aligned}$$


U(z̄+1)
U(z̄)
E[u]
U(z̄-1)
z̄-1
z̄
z+1

- People "dislike" risk: utility of a fair game is less than utility of having the expected value of the game for sure.
- To induce people to substitute to a risky alternative we must compensate them by making the riskless alternative unfair (by giving less than its expected value)
- For example, take $U(C) = \sqrt{C}$, $C_{low} = 4$ with $p = 1/2$ and $C_{high} = 16$ with $q = 1/2$:

$$\begin{aligned} E[C] &= \frac{1}{2}4 + \frac{1}{2}16 = 10 \\ E[U] &= \frac{1}{2}\sqrt{4} + \frac{1}{2}\sqrt{16} = 3 \\ U(E[C]) &= \sqrt{10} = 3.16 \end{aligned}$$

The person would be indifferent between this lottery and receiving $C = 9$ with certainty ($U(9) = 3$).

- The values of p and q that make the person indifferent between the sure thing and the risky alternative depend on:
  - The values of the possible outcomes (i.e., $\bar{C}+1$, $\bar{C}+2$, etc.)
  - The curvature of the utility function.

3.2.2 Measures of curvature

- Absolute risk aversion (ARA):

$$ARA \equiv -\frac{U''}{U'} > 0$$

- Relative risk aversion (RRA):

$$RRA \equiv -\frac{U''C}{U'} > 0$$

- Constant risk aversion utility functions:
  - Constant Absolute Risk Aversion (CARA): $U(C) = (1 - e^{-\beta C})$ with $\beta > 0$.

$$
\begin{aligned}
U' &= \beta e^{-\beta C} > 0 \\
U'' &= -\beta^2 e^{-\beta C} < 0 \\
ARA &= -\frac{-\beta^2 e^{-\beta C}}{\beta e^{-\beta C}} = \beta
\end{aligned}
$$

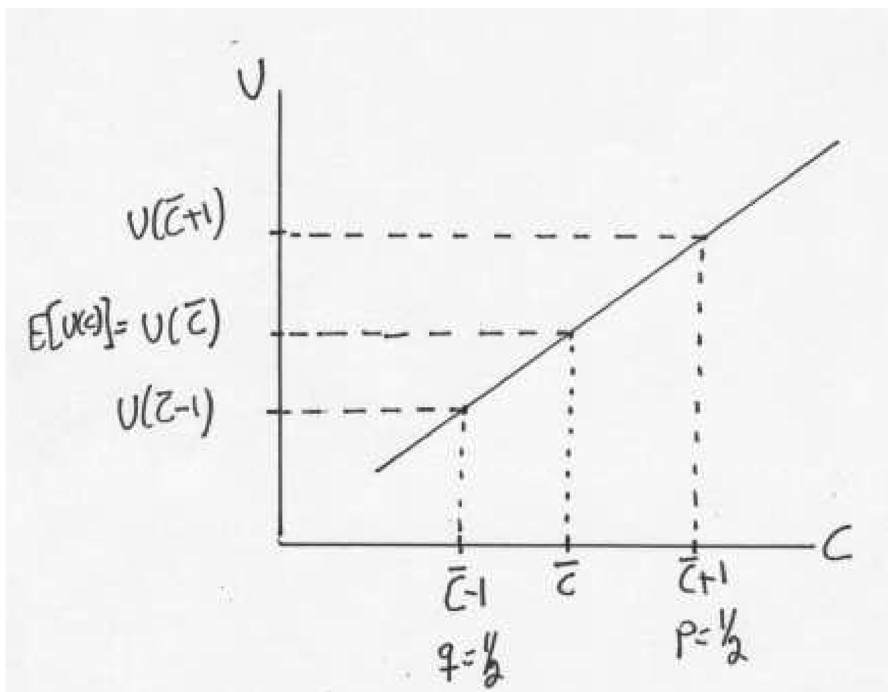
– Constant Relative Risk Aversion (CRRA): $U(C) = \frac{C^{1-\beta}-1}{1-\beta}$ with $\beta > 0$.

$$
\begin{aligned}
U' &= C^{-\beta} > 0 \\
U'' &= -\beta C^{-\beta-1} < 0 \\
RRA &= -\frac{-\beta C^{-\beta-1} C}{C^{-\beta}} = \beta
\end{aligned}
$$

3.3 Risk neutrality

1. Risk aversion arises from the fact that $U'' < 0$, which makes U concave.

2. If U were linear, then the consumer would be indifferent between a sure thing and a risky alternative with an expected value equal to the sure thing.



Linear utility:

$$\begin{aligned} U(C) &= \alpha C \\ E\left[U(C)\right] &= \frac{1}{2}[\alpha(\bar{C}-1)] + \frac{1}{2}[\alpha(\bar{C}+1)] \\ &= \alpha\bar{C} \\ &= U\left(E[C]\right) \end{aligned}$$

3. **Risk neutrality**

- A person indifferent to risk is said to be risk neutral
- Risk neutrality $\Leftrightarrow$ linear utility

4. **Risk-neutral probabilities**
   (a) Risk-neutral probabilities are the probabilities that would be consistent with **observed** prices if agents were risk-neutral instead of risk-averse.
   (b) Simple example
      i. Suppose there is a lottery ticket that pays 4 with probability $p = 1/2$ and 16 with probability $(1 - p) = 1/2$. What is the ticket worth?

ii. Suppose someone has the utility function $U(C) = \sqrt{C}$

- Then the expected utility of the lottery ticket is

$$\begin{aligned} E[U(C)] &= pU(C_{low}) + (1-p)U(C_{high}) \\ &= \frac{1}{2}\sqrt{4} + \frac{1}{2}\sqrt{16} \\ &= 3 \end{aligned}$$

- What value of C for sure (i.e, with probability one) gives the same utility?

$$\sqrt{C_S} = 3 \;\Leftrightarrow\; C_S = 9$$

which is the value of the lottery ticket since the person is indifferent between having 9 units of C for sure or the lottery ticket with the specified payoffs and probabilities. At that price, the ticket is a “fair game”.

iii. Now consider someone who is risk-neutral, with the simple linear utility function $U(C) = C$.

- What probabilities would make him willing to pay exactly 9 for the lottery ticket?
- Note that, for this person

  A. $U(9) = 9$

  B. $E[U(\text{lottery ticket})] = p^\star(4) + (1 - p^\star)(16) = 16 - 12p^\star$
- We want to find $p^\star$ so that

$$9 = 16 - 12p^\star$$

$\Rightarrow p^\star = 7/12$

$\Rightarrow$ With $p^\star = 7/12$ we have $E^\star[C] = 9$.