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AN EFFICIENT APPROACH TO PROBABILISTIC UNCERTAINTY ANALYSIS IN SIMULATION-BASED MULTIDISCIPLINARY DESIGN

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Abstract

In this paper , computationally efficient techniques for propagating the effect of uncertainty are developed to accommodate generic probabilistic representations of uncertain parameters and error estimation models in a multidisciplinary design system. To improve the computational efficiency of probabilistic uncertainty propagation in the context of highly coupled analyses, the first order sensitivity analysis and the moment matching method are employed. This is implemented by two techniques, namely, the system uncertainty analysis method (SUAM) and the concurrent subsystem uncertainty analysis method (CSSUAM). Depending on the number of variables and the number of disciplines involved, the effectiveness of these techniques varies. A mathematical example and an electronic packaging problem are used to verify the effectiveness of these approaches.

Nomenclature

CSSUAM	Concurrent Subsystem Uncertainty Analysis Method
F	simulation model
MCS	Monte Carlo Simulation
SUAM	System Uncertainty Analysis Method
x_i	Input variable of subsystem i
$\bar{x}_i$	mean value of x_i
x_s	sharing variable
$\bar{x}_s$	mean value of x_s
y	linking variable
$\bar{y}$	Mean value y
z	system output
$\bar{z}$	Mean value of z
e	model error
$\bar{e}$	mean value of model error e
s	standard deviation
s^2	Variance

1. Introduction

With the rapid advancement of computing technology, mathematical modeling and computer simulations have become common and standard methods for studying the behavior of complex systems. Since all the mathematical models and simulation models are only abstractions of the realities, the model-predicted performance and the actual system performance often deviate at certain levels. As a result, uncertainty always exists in simulation-based design. Under a multidisciplinary design environment, a system is composed of multidisciplinary subsystems each using a variety of disciplinary models with uncertainties associated with performance predications. These subsystems are often highly coupled where the performance prediction of one discipline may become the input of another discipline and vice versa [1, 2]. A critical issue in simulation-based multidisciplinary design is that the uncertainties of one discipline may be propagated to another discipline through the linking variables and the final output from the integrated multidisciplinary system has an accumulation of the uncertainties from the individual disciplines. It is important to study the impact of the uncertainty associated with simulation predictions on the analyses of individual subsystems as well as that of the whole

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system in a simple and efficient manner. This will become an important part of requirements tracking and design coordination.

Many researches on propagating the effect of uncertainty have been focused on design problems with a single discipline. Existing methods include Monte Carlo simulation[3], first-order second-moment analysis[4, 5], stochastic response surface method[6], reliability analysis based approaches[7], and parametric uncertainty analysis[8]. Most of these applications are limited to modeling the uncertainty of simulation input parameters. In recent developments, some preliminary results of propagating model (structure) uncertainty are reported. Du et al.[9] applied the extreme condition approach and the statistical approach to propagating the effect of both parameter and model uncertainties. The extreme condition approach is to derive the range of a system output in terms of the range of uncertainties by sub-optimizations (minimization and maximization of the performance). The statistical approach relies heavily on the use of data sampling to generate probabilistic distributions of system output. Very few works exist on propagating the effect of uncertainty in the context of multidisciplinary design. Preliminary results were published recently [10] on this topic. With their approach, model uncertainty is denoted by a range (bias) of the system output; the "worst case" concept and the first-order sensitivity analysis are used to evaluate the interval of the end performance of a multidisciplinary system.

Our aim in this paper is to develop computationally techniques for propagating the effect of uncertainty that could accommodate generic probabilistic representations of uncertain parameters and error estimation models in a multidisciplinary design system. To improve the computational efficiency of propagating the effect of uncertainty in the context of highly coupled analyses, the first order sensitivity analysis and the moment matching method are employed. This is implemented by two techniques, namely, the system uncertainty analysis method (SUAM) and the concurrent subsystem uncertainty analysis method (CSSUAM). A mathematical example and an electronic packaging problem are used to verify the effectiveness of these approaches. Observations are made on the effectiveness of the proposed methods and its relation to the number of linking variables and the number of disciplines in a multidisciplinary design system.

2. Uncertainty Modeling in Simulation-Based Design

When we describe a simple situation where we have a simulation model, representing a mapping from input variables **x** to output variables *z*, i.e., $z = F(\mathbf{x}, \mathbf{p})$, where **p** is the model parameter for a given fixed model structure $F(\cdot)$. Omitting the algorithmic errors related to computer implementations, several general sources contribute to the uncertainties in simulation predictions:

- variability of input values **x** (including both design parameters and design variables), called "*input parameter uncertainty*" [9],
- uncertainty due to limited information in estimating the characteristics of model parameters **p**, *called "model parameter uncertainty*" [11, 12], and
- uncertainty in the model structure $F(\cdot)$ itself (including uncertainty in the validity of the assumptions underlying the model), referred to as "*model structure uncertainty*" [13, 14].

We refer to "input parameter uncertainty" and "model parameter uncertainty" together as "*parameter uncertainty*", and "model structure uncertainty" as "*model uncertainty*". In this study, *we place uncertainties in simulation-based design within the broader context of prediction error.* If q is used for a target value representing truth or reality to be predicted by a simulation, prediction error refers to the difference $z - q$ and has components of *precision* (influenced by parameter uncertainty) and *accuracy* (due to model uncertainty). When studying accuracy, in many cases, we do not even know the accurate responses θ (realities) of the system or we cannot afford to test a wide range of behaviors, therefore we can only *estimate* the error. Under these circumstances, we could provide the estimation under some confidence levels by engineering judgement and describe the model uncertainty using probabilistic means, such as confidence levels associated with uncertain intervals. *We use this probabilistic notion to suggest that both the precision and accuracy components of uncertainties studied in this work will be measured by probability distributions of simulation predictions.*

Quantification of model uncertainty is more complicated compared to that of the parameter uncertainty. It is still an ongoing research issue in both academia and industry [13, 14]. One method to consider the model uncertainty is to introduce a function $e(\mathbf{x})$ of simulation input into the simulation output function $F(\mathbf{x})$ as

$$\mathbf{z} = F(\mathbf{x}) + e(\mathbf{x}) \tag{1}$$

Generally, $e(\mathbf{x})$ is a random function even under the condition that $\mathbf{x}$ is deterministic.

3. Techniques for Probabilistic Uncertainty Analysis in Multidisciplinary Design

3.1 Description of Multidisciplinary System under Uncertainties

The techniques for probabilistic uncertainty analysis are developed in this work for the n-discipline system shown in Fig.1, where each box represents a simulation program that belongs to a discipline (or subsystem) for design evaluation. $\mathbf{x}_s$ are the input variables considered by more than one subsystem[1]; $\mathbf{x}_s$ are also called sharing variables in the literature [15]. $\mathbf{x}_i$ $(\mathrm{i}=1,\mathrm{n})$ are the input variables of subsystem *i*.

Note that $\mathbf{x}_s$ and $\mathbf{x}_i$ are mutually exclusive sets of input variables. In general, $\mathbf{x}_s$ and $\mathbf{x}_i$ have uncertainties associated with them which can be expressed by probabilistic distributions. We assume that all the elements both in $\mathbf{x}_s$ and $\mathbf{x}_i$ are mutually independent.

$\mathbf{y}_{ij}(i \neq j)$ are linking variables, which are those functional outputs calculated in subsystem *i* , at the same time, are required as inputs to subsystem *j*. For simplification of representation, we denote $\mathbf{y}_i = \{\mathbf{y}_{ij} \,|\, j=1,n,\, j \neq i\}$ as the set of linking variables generated as outputs from subsystem *i* and taken as inputs to the other subsystems and $\mathbf{y}^i = \{\mathbf{y}_1, \cdots \mathbf{y}_{i-1}, \mathbf{y}_{i+1}, \cdots, \mathbf{y}_n\}$ as the set of linking variables generated as outputs from each of the subsystem except subsystem *i* and taken as inputs to subsystem *i*.

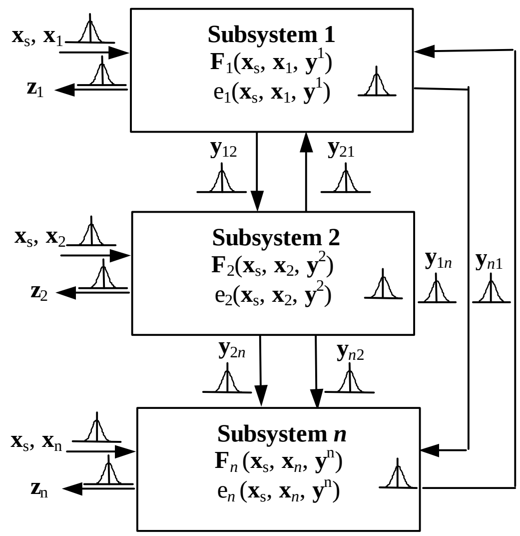
For subsystem *i*, based on the subsystem simulation model $\mathbf{F}_{yi}(\cdot)$ and the corresponding error model $\mathbf{e}_{yi}(\cdot)$, the linking variables can be derived as:

$$\mathbf{y}_i = \mathbf{F}_{yi}(\mathbf{x}_s, \mathbf{x}_i, \mathbf{y}^i) + \mathbf{e}_{yi}(\mathbf{x}_s, \mathbf{x}_i, \mathbf{y}^i)\,. \qquad (2)$$

Similarly, the general output of subsystem *i*, $\mathbf{z}_i(\mathrm{i}=1,\mathrm{n})$, can be derived as:

$$\mathbf{z}_i = \mathbf{F}_{zi}(\mathbf{x}_s, \mathbf{x}_i, \mathbf{y}^i) + \mathbf{e}_{zi}(\mathbf{x}_s, \mathbf{x}_i, \mathbf{y}^i)\,. \qquad (3)$$

The general outputs of each subsystem $\mathbf{z}_i$, which may include linking variables, are often associated with the system attributes for the evaluations of constraints and objectives in optimization. For simplification, we assume all the error models $\mathbf{e}_{yi}(\cdot)$ and $\mathbf{e}_{zi}(\cdot)$ are the functions of the mean values $\bar{\mathbf{x}}_s, \bar{\mathbf{x}}_i, \bar{\mathbf{y}}^i$.



$\mathbf{F}_i=\{\mathbf{F}_{zi}, \mathbf{F}_{yij}, i\neq j\}$, $e_i=\{e_{zi}, e_{yij}, i\neq j\}$

Figure 1 Coupled system

As discussed in Section 2, using the probabilistic notion, the error models $\mathbf{e}_{yi}$ and $\mathbf{e}_{zi}$ are represented by random functions and the system inputs $\mathbf{x}_s$ and $\mathbf{x}_i$ are described by random variables. As a result, the linking variables $\mathbf{y}_i$ and the system outputs $\mathbf{z}_i$ are random variables with distributions. In propagating the effect of uncertainty, the goal is to quantify the distributions of system outputs $\mathbf{z}_i$ for the given parameter uncertainty and the model uncertainty. If we choose to use the first and second moments (mean value and variance) to describe the distributions, the problem can be states as:

Given:	mean values and variances of input variables
	$\bar{\mathbf{x}}_s$, $\bar{\mathbf{x}}_i$, $\mathbf{s}_{xs}$, and $\mathbf{s}_{xi}$ $\quad(\mathrm{i}=1,\mathrm{n})$
	mean values and variances of error models
	$\bar{\mathbf{e}}_{yi}$, $\bar{\mathbf{e}}_{zi}$, $\mathbf{s}_{eyi}$ and $\mathbf{s}_{ezi}$ $\quad(\mathrm{i}=1,\mathrm{n})$
Find:	mean values and variances of system outputs
	$\bar{\mathbf{z}}_i$ and $\mathbf{s}_{zi}$ $\quad(\mathrm{i}=1,\mathrm{n})$

Both the system uncertainty analysis method (SUAM) and the concurrent subsystem uncertainty analysis method (CSSUAM) are developed to derive the mean and the variance of a system attribute.

[1] All the bold letters are used to denote both variable vectors and function vector.

3.2 System Uncertainty Analysis Method (SUAM)

The SUAM is an approach that utilizes Taylor approximations as well as global sensitivity analysis (first-order derivatives) to evaluate the mean and variance of a system attribute subject to both parameter and model uncertainties in a multidisciplinary system.

From Eqns. (2) and (3), the mean values of linking variables and system outputs can be approximated as

$$\bar{\mathbf{y}}_i = \mathbf{F}_{yi}(\bar{\mathbf{x}}_s, \bar{\mathbf{x}}_i, \bar{\mathbf{y}}^i) + \bar{\mathbf{e}}_{yi} \tag{4}$$

$$\bar{\mathbf{z}}_i = \mathbf{F}_{zi}(\bar{\mathbf{x}}_s, \bar{\mathbf{x}}_i, \bar{\mathbf{y}}^i) + \bar{\mathbf{e}}_{zi} \tag{5}$$

The evaluations of Eqns. (4) and (5) require analyses at the system level.

To obtain the variances of system outputs with the SUAM, first, linking variables $\mathbf{y}_i$ $(\mathrm{i}=1,\mathrm{n})$ are linearized by the first-order Taylor approximations at the system level and derived simultaneously based on a set of linear equations. Then, within each individual subsystem, the variances of system outputs are obtained based on subsystem analyses. These are further explained as the following.

From Eqn. (2), the linking variables $\mathbf{y}_i$ are approximated using Taylor's expansion as

$$\Delta\mathbf{y}_i = \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{\partial \mathbf{F}_{yi}}{\partial \mathbf{y}_j}\Delta\mathbf{y}_j + \frac{\partial \mathbf{F}_{yi}}{\partial \mathbf{x}_s}\Delta\mathbf{x}_s + \frac{\partial \mathbf{F}_{yi}}{\partial \mathbf{x}_i}\Delta\mathbf{x}_i + \Delta\mathbf{e}_{yi} \quad \text{(i=1, n)}, \tag{6}$$

which yields

$$\mathbf{A}\Delta\mathbf{y} = \mathbf{B}\Delta\mathbf{x}_s + \mathbf{C}\Delta\mathbf{x} + \mathbf{D}, \tag{7}$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{I}_1 & -\frac{\partial \mathbf{F}_{y1}}{\partial \mathbf{y}_2} & \cdots & -\frac{\partial \mathbf{F}_{y1}}{\partial \mathbf{y}_n} \\ -\frac{\partial \mathbf{F}_{y2}}{\partial \mathbf{y}_1} & \mathbf{I}_2 & \cdots & -\frac{\partial \mathbf{F}_{y2}}{\partial \mathbf{y}_n} \\ \cdots & \cdots & \cdots & \cdots \\ -\frac{\partial \mathbf{F}_{yn}}{\partial \mathbf{y}_1} & -\frac{\partial \mathbf{F}_{yn}}{\partial \mathbf{y}_2} & \cdots & \mathbf{I}_n \end{bmatrix}$$

($\mathbf{I}_i$ $(\mathrm{i}=1,\mathrm{n})$ are the identity matrixes),

$$\mathbf{B} = \begin{bmatrix} \frac{\partial \mathbf{F}_{y1}}{\partial \mathbf{x}_s} \\ \frac{\partial \mathbf{F}_{y2}}{\partial \mathbf{x}_s} \\ \cdots \\ \frac{\partial \mathbf{F}_{yn}}{\partial \mathbf{x}_s} \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} \frac{\partial \mathbf{F}_{y1}}{\partial \mathbf{x}_1} & 0 & \cdots & 0 \\ 0 & \frac{\partial \mathbf{F}_{y2}}{\partial \mathbf{x}_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \frac{\partial \mathbf{F}_{yn}}{\partial \mathbf{x}_n} \end{bmatrix},$$

$$\mathbf{D} = \begin{bmatrix} \mathbf{e}_{y1} - \bar{\mathbf{e}}_{y1} \\ \mathbf{e}_{y2} - \bar{\mathbf{e}}_{y2} \\ \cdots \\ \mathbf{e}_{yn} - \bar{\mathbf{e}}_{yn} \end{bmatrix}$$

$$\Delta\mathbf{x}_s = \mathbf{x}_s - \bar{\mathbf{x}}_s, \quad \Delta\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 - \bar{\mathbf{x}}_1 \\ \mathbf{x}_2 - \bar{\mathbf{x}}_2 \\ \cdots \\ \mathbf{x}_n - \bar{\mathbf{x}}_n \end{bmatrix}, \text{ and}$$

$$\Delta\mathbf{y} = \begin{bmatrix} \mathbf{y}_1 - \bar{\mathbf{y}}_1 \\ \mathbf{y}_2 - \bar{\mathbf{y}}_2 \\ \cdots \\ \mathbf{y}_n - \bar{\mathbf{y}}_n \end{bmatrix}.$$

Solving Eqn. (7), we have

$$\Delta\mathbf{y} = \mathbf{A}^{-1}\mathbf{B}\Delta\mathbf{x}_s + \mathbf{A}^{-1}\mathbf{C}\Delta\mathbf{x} + \mathbf{A}^{-1}\mathbf{D}. \tag{8}$$

Similarly, we can approximate the general system outputs as

$$\Delta\mathbf{z}_i = \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{\partial \mathbf{F}_{zi}}{\partial \mathbf{y}_j}\Delta\mathbf{y}_j + \frac{\partial \mathbf{F}_{zi}}{\partial \mathbf{x}_s}\Delta\mathbf{x}_s + \frac{\partial \mathbf{F}_{zi}}{\partial \mathbf{x}_i}\Delta\mathbf{x}_i + \Delta\mathbf{e}_{zi} \quad \text{(i=1, n).} \tag{9}$$

The reorganization of the above equation yields

$$\begin{aligned} \Delta\mathbf{z} &= \mathbf{E}?\,\mathbf{y} + \mathbf{F}\Delta\mathbf{x}_s + \mathbf{G}\Delta\mathbf{x} + \mathbf{H} \\ &= [\mathbf{E}(\mathbf{A}^{-1}\mathbf{B}) + \mathbf{F}]\Delta\mathbf{x}_s + [\mathbf{E}(\mathbf{A}^{-1}\mathbf{C}) + \mathbf{G}]\Delta\mathbf{x} \\ &\quad + \mathbf{E}\mathbf{A}^{-1}\mathbf{D} + \mathbf{H}, \end{aligned} \tag{10}$$

where

$$\mathbf{E} = \begin{bmatrix} 0 & \frac{\partial \mathbf{F}_{z1}}{\partial \mathbf{y}_2} & \cdots & \frac{\partial \mathbf{F}_{z1}}{\partial \mathbf{y}_n} \\ \frac{\partial \mathbf{F}_{z2}}{\partial \mathbf{y}_1} & 0 & \cdots & \frac{\partial \mathbf{F}_{z2}}{\partial \mathbf{y}_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \mathbf{F}_{zn}}{\partial \mathbf{y}_1} & \frac{\partial \mathbf{F}_{zn}}{\partial \mathbf{y}_2} & \cdots & 0 \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} \frac{\partial \mathbf{F}_{z1}}{\partial \mathbf{x}_s} \\ \frac{\partial \mathbf{F}_{z2}}{\partial \mathbf{x}_s} \\ \cdots \\ \frac{\partial \mathbf{F}_{zn}}{\partial \mathbf{x}_s} \end{bmatrix},$$

$$\mathbf{G}=\begin{bmatrix}\frac{\partial \mathbf{F}_{z1}}{\partial \mathbf{x}_1} & 0 & \cdots & 0\\ 0 & \frac{\partial \mathbf{F}_{z2}}{\partial \mathbf{x}_2} & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \cdots & \frac{\partial \mathbf{F}_{zn}}{\partial \mathbf{x}_n}\end{bmatrix},\ \Delta\mathbf{z}=\begin{bmatrix}\mathbf{z}_1-\bar{\mathbf{z}}_1\\ \mathbf{z}_2-\bar{\mathbf{z}}_2\\ \cdots\\ \mathbf{z}_n-\bar{\mathbf{z}}_n\end{bmatrix},$$

and $\mathbf{H}=\begin{bmatrix}\mathbf{e}_{z1}-\bar{\mathbf{e}}_{z1}\\ \mathbf{e}_{z2}-\bar{\mathbf{e}}_{z2}\\ \cdots\\ \mathbf{e}_{zn}-\bar{\mathbf{e}}_{zn}\end{bmatrix}$.

Since $\Delta\mathbf{x}_s$, $\Delta\mathbf{x}$, $\mathbf{D}$ and $\mathbf{H}$ in Eqns. (8) and (10) are mutually independent, the variance of a general system output can be expressed as

$$\mathbf{D}_z=\mathbf{ID}_{xs}+\mathbf{JD}_x+\mathbf{KD}_{ye}+\mathbf{D}_{ze}, \qquad (11)$$

where

$$\mathbf{D}_z=\begin{bmatrix}\mathbf{s}^2_{z1}\\ \mathbf{s}^2_{z2}\\ \cdots\\ \mathbf{s}^2_{zn}\end{bmatrix},\ \mathbf{D}_y=\begin{bmatrix}\mathbf{s}^2_{y1}\\ \mathbf{s}^2_{y2}\\ \cdots\\ \mathbf{s}^2_{yn}\end{bmatrix},\ \mathbf{D}_x=\mathbf{s}^2_x,\ \mathbf{D}_{xs}=\begin{bmatrix}\mathbf{s}^2_{x1}\\ \mathbf{s}^2_{x2}\\ \cdots\\ \mathbf{s}^2_{xn}\end{bmatrix},$$

$$\mathbf{D}_{ye}=\begin{bmatrix}\mathbf{s}^2_{ye1}\\ \mathbf{s}^2_{ye2}\\ \cdots\\ \mathbf{s}^2_{yen}\end{bmatrix},\ \mathbf{D}_{ze}=\begin{bmatrix}\mathbf{s}^2_{ze1}\\ \mathbf{s}^2_{ze2}\\ \cdots\\ \mathbf{s}^2_{zen}\end{bmatrix},$$

$$\mathbf{I}=\{i_{ij}\},\ i_{ij}=\{\mathbf{E}(\mathbf{A}^{-1}\mathbf{B})+\mathbf{F}\}^2_{ij},$$

$$\mathbf{J}=\{j_{ij}\},\ j_{ij}=\{\mathbf{E}(\mathbf{A}^{-1}\mathbf{C})+\mathbf{G}\}^2_{ij}$$

$$\mathbf{K}=\{k_{ij}\},\ k_{ij}=\{\mathbf{EA}^{-1}\}^2_{ij},$$

and all the $\mathbf{s}^2$ are the variance vectors.

From Eqn. (11), it is noted that with this approach the total variation of a system output can be derived as the sum of the variations contributed by four individual sources. Each term on the right hand side of Eqn. (11) represents the variation contributed by the uncertainties of the sharing variables $\mathbf{x}_s$, the variation of subsystem input variables $\mathbf{x}_i$, variation of linking variable $\mathbf{y}_i$ due to model uncertainty, and the variation of system output $\mathbf{z}_i$ due to model uncertainty, respectively.

3.3 Concurrent Subsystem Uncertainty Analysis Method (CSSUAM)

With the SAUM approach, the sensitivity analysis and solution search of simultaneous linear equations need to be conducted at the system level. To reduce the computational demand of system level analysis and to make use of the parallel computing technique, we propose a concurrent subsystem uncertainty analysis method (CSSUAM) for uncertainty propagation. The method will facilitate the parallelization of the variance evaluation for system outputs that are contributed by different subsystems. The method for propagating the effect of uncertainty is developed in such a way that the mean and variance of each of the subsystems can be calculated simultaneously by the parallel computing approach. Here the compatibility among the subsystems (the match of linking variables $\mathbf{y}_i$) should be taken into account. To match the random linking variables $\mathbf{y}_i$, we should consider the matches of both the mean values $\bar{\mathbf{y}}_i$ and variances $\mathbf{s}_{yi}$. As shown in Figure 2, the compatibility is achieved by a system level optimizier which sets the target values of the mean and variance of the linking variables and minimizes the deviations between the targets and those that are actually generated through the subsystems analyses. The idea can be generated as the following unconstrained optimization model:

> **Given**: mean values and variances of input variables
> $\bar{\mathbf{x}}_s$, $\bar{\mathbf{x}}_i$, $\mathbf{s}_{xs}$, and $\mathbf{s}_{xi}$ $(\mathrm{i}=1,\mathrm{n})$
> mean values and variances of error models
> $\bar{\mathbf{e}}_{yi}$, $\bar{\mathbf{e}}_{zi}$, $\mathbf{s}_{eyi}$ and $\mathbf{s}_{ezi}$ $(\mathrm{i}=1,\mathrm{n})$
> **Find**: mean values and variances of linking variables
> $\bar{\mathbf{y}}_i$ and $\mathbf{s}_{yi}$ $(i=1,n)$
> **Minimize**: $d=\min\sum_{i=1}^{n}[(\bar{\mathbf{y}}_i-\bar{\mathbf{y}}^*_i)^2+(\mathbf{s}_{yi}-\bar{\mathbf{s}}^*_{yi})^2]$

$\bar{\mathbf{y}}^*_i$ and $\bar{\mathbf{s}}^*_{yi}$ (i=1, n) are the mean values and the standard deviations of linking variables $\mathbf{y}_i$. $\bar{\mathbf{y}}^*_i$ and $\bar{\mathbf{s}}^*_{yi}$ can be calculated within subsystems *i* based on the input variables $\mathbf{x}_s$, $\mathbf{x}_i$ (described by $\bar{\mathbf{x}}_s$, $\bar{\mathbf{x}}_i$ and $\mathbf{s}_{xs}$, $\mathbf{s}_{xi}$) and linking variables $\mathbf{y}^i$ (described by $\bar{\mathbf{y}}^i$ and $\mathbf{s}_{y^i}$).

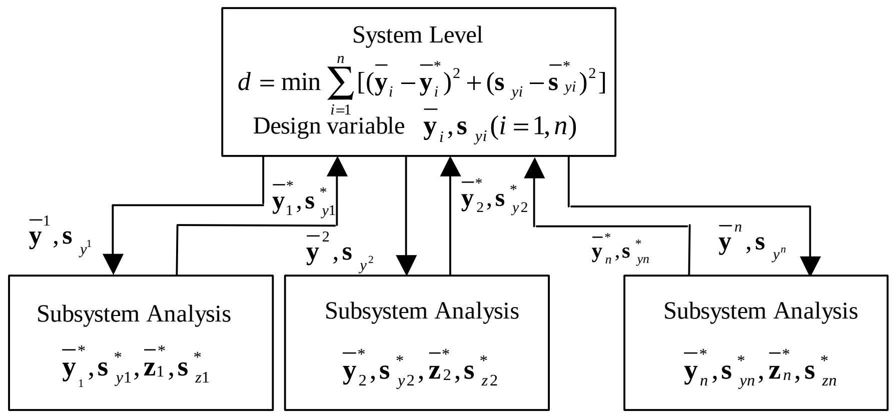


Figure 2 Concurrent Subsystem Uncertainty Analysis

$\bar{\mathbf{y}}_i^*$ can be calculated by

$$\bar{\mathbf{y}}_i^* = \mathbf{F}_{yi}(\bar{\mathbf{x}}_s, \bar{\mathbf{x}}_i, \bar{\mathbf{y}}^i) + \bar{\mathbf{e}}_{yi} \quad (12)$$

If neglecting the dependence between $\mathbf{y}_i$ and $\mathbf{x}_s, \mathbf{x}_i$, $\mathbf{y}_j (j \neq i)$, based on Eqn. (2), the variance $\bar{\mathbf{s}}_{yi}^*$ of $\mathbf{y}_i$ is obtained as

$$\mathbf{s}_{yi}^{*2} = \sum_{\substack{j=1 \\ j \neq i}}^{n} \left(\frac{\partial \mathbf{F}_{yi}}{\partial \mathbf{y}_j} \right)^2 \mathbf{s}_{yj}^2 + \left(\frac{\partial \mathbf{F}_{yi}}{\partial \mathbf{x}_s} \right)^2 \mathbf{s}_{xs}^2 + \left(\frac{\partial \mathbf{F}_{yi}}{\partial \mathbf{x}_i} \right)^2 \mathbf{s}_{xi}^2 + \mathbf{s}_{eyi}^2 \quad (13)$$

Once the mean values and variances of $\mathbf{y}_i$ are obtained using the above optimization model, the mean value $\bar{\mathbf{z}}_i$ can be determined based on Eqn. (3). If neglecting the dependence between $\mathbf{z}_i$ and $\mathbf{x}_s, \mathbf{x}_i$, $\mathbf{y}_i$, based on Eqn. (9), the variance of $\mathbf{z}_i$ can be obtained as:

$$\mathbf{s}_{zi}^2 = \sum_{j=1}^{n} \left(\frac{\partial \mathbf{F}_{zi}}{\partial \mathbf{y}_j} \right)^2 \mathbf{s}_{yj}^2 + \left(\frac{\partial \mathbf{F}_{zi}}{\partial \mathbf{x}} \right)^2 \mathbf{s}_{xs}^2 + \left(\frac{\partial \mathbf{F}_{zi}}{\partial \mathbf{x}_i} \right)^2 \mathbf{s}_{xi}^2 + \mathbf{s}_{ezi}^2 \quad (14)$$

4. Examples

4.1 A Mathematical Example

To verify the proposed techniques, results generated using a Monte Carlo Simulation (MCS) procedure (Figure 3) are considered as the correct results and compared with those from the SUAM and CSSUAM. The strategy of MCS is developed as shown in Figure 3.

For the example problem with two subsystems (Fig. 4), the functional relationships are represented as:

$\mathbf{x}_s = \{x_1\}$, $\mathbf{x}_1 = \{x_2, x_3\}$, $\mathbf{x}_2 = \{x_4, x_5\}$

$\mathbf{y}_1 = \mathbf{y}_{12} = \{y_{12}\}$, $\mathbf{y}_2 = \mathbf{y}_{21} = \{y_{21}\}$

$\mathbf{z}_1 = \{z_1\}$, $\mathbf{z}_2 = \{z_2\}$

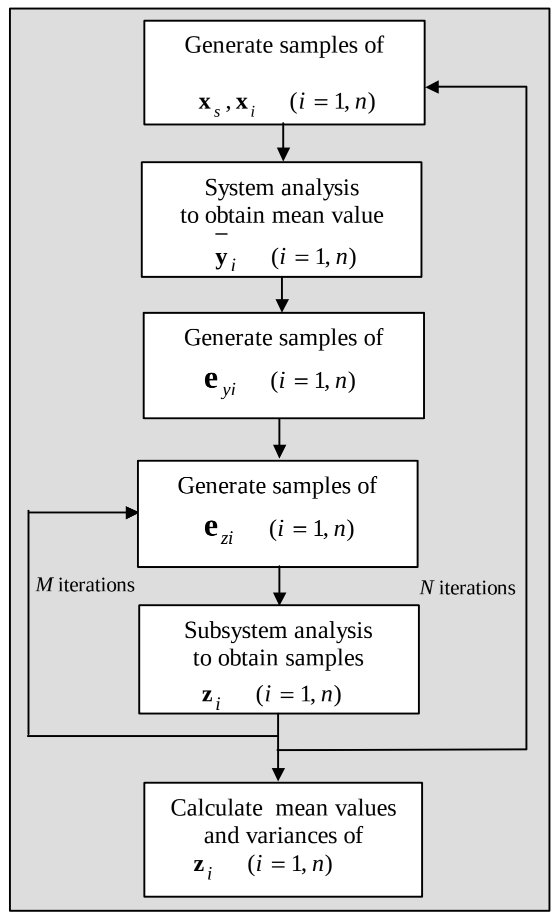


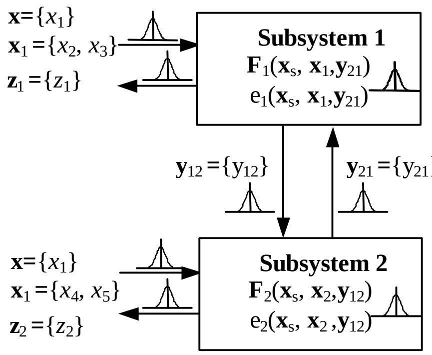
Figure 3 Procedure of Monte Carlo Simulation

$\mathbf{e}_{y1}(\mathbf{x}_s, \mathbf{x}_1, \mathbf{y}_1) = \{e_{y12}(\mathbf{x}_s, \mathbf{x}_1, \mathbf{y}_2)\}$

$\mathbf{e}_{y2}(\mathbf{x}_s, \mathbf{x}_2, \mathbf{y}_2) = \{e_{y21}(\mathbf{x}_s, \mathbf{x}_2, \mathbf{y}_1)\}$

$\mathbf{e}_{z1}(\mathbf{x}_s, \mathbf{x}_1, \mathbf{y}_1) = \{e_{z1}(\mathbf{x}_s, \mathbf{x}_1, \mathbf{y}_1)\}$

$$\mathbf{e}_{z2}(\mathbf{x}_s,\mathbf{x}_2,\mathbf{y}_2)=\{e_{z2}(\mathbf{x}_s,\mathbf{x}_2,\mathbf{y}_2)\}$$
$$\mathbf{F}_{y12}(\mathbf{x}_s,\mathbf{x}_1,\mathbf{y}_1)=\{F_{y12}(\mathbf{x}_s,\mathbf{x}_1,\mathbf{y}_1)\}$$
$$=x_1^2+2x_2-x_3+2\sqrt{y_{21}}$$
$$\mathbf{F}_{z1}(\mathbf{x}_s,\mathbf{x}_1,\mathbf{y}_1)=\{F_{z1}(\mathbf{x}_s,\mathbf{x}_1,\mathbf{y}_1)\}$$
$$=x_1^2+2x_2+x_3+x_2e^{-y_{21}}$$
$$\mathbf{F}_{y21}(\mathbf{x}_s,\mathbf{x}_2,\mathbf{y}_2)=\{F_{y21}(\mathbf{x}_s,\mathbf{x}_2,\mathbf{y}_2)\}$$
$$=x_1x_4+x_4^2+x_5+y_{12}$$
$$\mathbf{F}_{z2}(\mathbf{x}_s,\mathbf{x}_2,\mathbf{y}_2)=\{F_{z2}\}=\sqrt{x_1}+x_4+x_5(0.4y_{12})$$



$\mathbf{F}_1=\{\mathbf{F}_{z1}, \mathbf{F}_{y12}\}$, $\mathbf{F}_2=\{\mathbf{F}_{z2},\mathbf{F}_{y21}\}$

Figure 4 Information Flow – Example

The parameter and model uncertainties are represented by distributions as defined in Table 1. The distributions of system output are generated using the SUAM and the CSSUAM. They are compared with the results from the MSC. It is observed that the mean values of system outputs using the SUAM, the CSSUAM, and the MSC are very close. For example, at the first design point, the mean values of z_1 obtained from the SUAM, the CSSUAM, and the MSC are 4.0, 4.0020, and 4.0323, respectively. The mean values of z_2 are 3.8867, 3.8765, 3.8940, respectively at the same point. Table 2 lists the comparisons of the variances of system outputs z_1 and z_2, where $M\times N=10^5$ is used as the simulation size for MCS.

Table 1. Descriptions of Parameter and Model Uncertainties

	Mean value	Standard deviation	Distribution
x_1	$\bar{x}_1$	$0.1\bar{x}_1$	Normal
x_2	$\bar{x}_2$	$0.1\bar{x}_2$	Normal
x_3	$\bar{x}_3$	$0.1\bar{x}_3$	Normal
x_4	$\bar{x}_4$	$0.1\bar{x}_4$	Normal
x_5	$\bar{x}_5$	$0.1\bar{x}_5$	Normal
e_{y12}	0	$0.1\bar{y}_{12}$	Normal
e_{y21}	0	$0.1\bar{y}_{21}$	Normal
e_{z1}	0	$0.1\bar{z}_1$	Normal
e_{z2}	0	$0.1\bar{z}_2$	Normal

From the comparison in Table 2, it is noted that the variance obtained using the SUAM are very close to that from the Monte Carlo simulation. For three sets of input values $(\bar{x}_1\sim\bar{x}_5)$, the errors of s_{z1} and s_{z2} are all less than 1%. The CSSUAM also provides a good approximation of the system variance, however it is not as accurate as SUAM. The reason is that the CSSUAM is based on the assumption of independence between $\mathbf{y}_i$ and $\mathbf{x}_s$, $\mathbf{x}_i$, $\mathbf{y}_j (j\neq i)$ and the assumption of independence between $\mathbf{z}_i$ and $\mathbf{x}_s$, $\mathbf{x}_i$, $\mathbf{y}_i$ (see Eqns. (13) and (14)), while correlations actually exist. Even so, the approximation is still acceptable from the practical viewpoint.

Table 2. Comparison of the Variances of System Outputs

$(\bar{x}_1,\bar{x}_2,\bar{x}_3,\bar{x}_4,\bar{x}_5)$	Method	s^2_{z1}	Error of s^2_{z1}	s^2_{z2}	Error of s^2_{z2}
(1,1,1,1)	SUAM	0.250	0.04%	0.2296	-0.86%
	CSSUAM	0.250	0.04%	0.2109	-8.94%
	MCS	0.2499	—	0.2316	—
(2,2,2,2,2)	SUAM	1.840	-0.49%	1.2292	-0.11%
	CSSUAM	1.840	-0.49%	1.1292	-8.23%
	MCS	1.8490	—	1.2305	—
(2,5,2,5,2)	SUAM	4.240	-0.4%	2.7553	0.11%
	CSSUAM	4.240	-0.4%	2.5936	-5.77%
	MCS	4.2572	—	2.7524	—

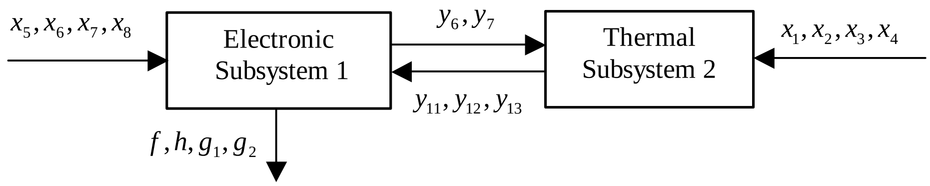


Figure 5 Electrical Package Problem

4.2 Electronic Packaging Problem

The electronic packaging problem (http://fmad-www.larc.nasa.gov/mdob/MDOB) is a bench-mark multidisciplinary problem comprising the coupling between electronic and thermal subsystems. Component resistances (in subsystem 1) are affected by operating temperatures in (subsystem 2), while the temperatures depend on the resistances. The system is demonstrated in Figure 5. A detailed problem statement is provided in Appendix.

The system analysis consists of the coupled thermal and electrical analyses. The component temperatures calculated in the thermal analysis are needed in the electrical analysis in order to compute the power dissipation of each resistor. Likewise, the power dissipation of each component must be known in order for the thermal analysis to compute the temperatures.

There are eight input variables x_1~ x_8, five linking variables y_6, y_7, y_{11}, y_{12}, y_{13}, and four system outputs f, h, g_1 and g_2. The sets of variables and functions in the two subsystems are as follows:

Thermal Analysis:

Input variables:

$\mathbf{x}_s = \{f\}$, $\mathbf{x}_1 = \{x_5, x_6, x_7, x_8\}$.

Linking variables:

$\mathbf{y}_1 = \mathbf{y}_{12} = \{y_{11}, y_{12}, y_{13}\}$.

System outputs:

$\mathbf{z}_1 = \{f, h, g_1, g_2\}$.

$\{f\}$ stands for an empty set.

Electronic Analysis:

Input variables:

$\mathbf{x}_s = \{f\}$, $\mathbf{x}_2 = \{x_1, x_2, x_3, x_4\}$.

Linking variables:

$\mathbf{y}_2 = \mathbf{y}_{21} = \{y_6, y_7\}$.

System outputs:

$\mathbf{z}_2 = \{f\}$.

Of the two subsystems, the thermal analysis is more complex. The thermal analysis requires a finite difference solution for the temperature distribution calculation. The remaining equations in the thermal subsystem are solved algebraically. All equations of the electrical system are solved algebraically.

The original electronic packaging problem involves only deterministic analyses where no uncertainty is considered. To illustrate the proposed probabilistic analysis, we assume uncertainties are associated with the input variables x_i (i = 1,2,…,8) and the thermal simulation models. All of them can be described by normal distributions.

The system outputs are calculated by the SUAM, the CSSUAM, and the MSC ($M \times N = 10^5$) at two design points. We set the standard deviations at about 10% of their relative mean values for the first design point. The second design point is different from the first one in its mean values while the standard deviations remain the same. The results are shown in Tables 3 and 4.

It is noted that the results from both the SUAM and the CSSUAM are good approximations of the first two moments (mean and variance) for system outputs in this problem.

Table 3. System Output at Design Point 1

Inputs: x_1~N(0.1, 0.01), x_2~N(0.1, 0.01), x_3~N(0.1, 0.001), x_4~N(0.05, 0.01), x_5~N(100, 10.0), x_6~N(0.004, 0.001), x_7~N(100, 10.0), x_8~N(0.004, 0.001), y_{11}~N($\bar{y}_{11}$,5.0), y_{12} ~N($\bar{y}_{12}$,5.0)

Outputs	Method	Mean Value	Standard Deviation
f	SUAM	-1847.2896	5188.5094
	CSSUAM	-1846.8066	5399.9184
	MCS	-1871.5695	5073.8715
h	SUAM	-2.1043×10^{-3}	1.2952×10^{-2}
	CSSUAM	2.6836×10^{-6}	1.3495×10^{-2}
	MCS	4.2534×10^{-5}	1.2615×10^{-2}
g_1	SUAM	-44.1101	4.9276
	CSSUAM	-44.1088	5.0016
	MCS	-44.0993	5.0366
g_2	SUAM	-44.0986	5.2155
	CSSUAM	-44.1010	5.0018
	MCS	-44.0319	5.0539

Note: N(· , ·) stands for normal distribution.

Table 4. System Output at Design Point 2

Inputs: x_1~N(0.08, 0.01), x_2~N(0.08, 0.01), x_3~N(0.055, 0.001), x_4~N(0.0275, 0.01), x_5~N(505, 10.0), x_6~N(0.0065, 0.001), x_7~N(505, 10.0), x_8~N(0.0065, 0.001), y_{11}~N($\overline{y}_{11}$,5.0), y_{12} ~N($\overline{y}_{12}$,5.0)			
Outputs	Method	Mean Value	Standard Deviation
f	SUAM	-1018.4123	428.4079
	CSSUAM	-1018.4960	389.6195
	MCS	-1027.7747	417.7766
h	SUAM	3.90×10^{-6}	9.7025×10^{-4}
	CSSUAM	3.4243×10^{-7}	9.7160×10^{-4}
	MCS	-1.1879×10^{-4}	9.3617×10^{-4}
g_1	SUAM	-48.8707	4.9669
	CSSUAM	-48.8741	5.0001
	MCS	-48.8182	4.9231
g_2	SUAM	-48.8691	5.0360
	CSSUAM	-48.8708	5.0001
	MCS	-48.7507	5.0905

4.3 Discussions

For the evaluation of the mean values of system outputs, only one analysis at the system level is required when using the SUAM. No analysis at the system level is required if using the CSSUAM. For the calculation of variance, if neglecting the evaluation of the first-order derivatives, SUAM only needs one analysis at the system level for solving the simultaneous linear equations (see Eqn. (7)) to obtain the linear approximations of linking variables $\Delta\mathbf{y}$. The amount of calculation depends on the dimension of the matrix **A** in Eqn. (7) which is determined by the number of linking variables $\mathbf{y}_{ij}\,(\mathrm{i}=1,\mathrm{n},\,\mathrm{j}=1,\mathrm{n},\mathrm{i}\neq\mathrm{j})$. This amount is also associated with the total number of subsystems involved. For the mathematical example problem, the dimension of the matrix **A** is 2 x 2, while in the electronic packaging problem, the dimension of the matrix **A** is 4 x 4. Once $\Delta\mathbf{y}$ is obtained, the remaining computations (subsystem analysis) can be executed within individual subsystems. If the first-order derivatives are evaluated by a numerical method, the additional analyses at the system level are needed. Since all the partial derivative of linking variables and system outputs with respect to design variable $\mathbf{x}_s$ and $\mathbf{x}_i$ (i=1, n) need to be calculated at the system level, the number of analysis at the system level for derivative evaluation is equal to the total number of design variables. This number is five in the mathematical example and eight in the electronic packaging problem. A small number of linking variables and design variables is preferred to deal with the inverse of inversion of matrix **A** and to conduct analysis at the system level

When the number of linking variables and the total number of input variables is large, to reduce the computational effort, the CSSUAM can be employed, which is more efficient for this situation. The method does not involve any system level analysis. All the calculations are implemented at the subsystems where the dimensions can be significantly reduced compared to the SSAM. On the other hand, the number of subsystem analysis may increase due to the optimization process at the top level (see Figure 2). In the mathematical example, even though no system analysis is involved, there are more than 50 subsystem calls in the optimization process (The number depends on the optimization algorithm and parameter settings, such as the starting point).

5. Conclusions

Two techniques, namely, the system uncertainty analysis method (SUAM) and the concurrent subsystem uncertainty analysis method (CSSUAM), are developed for improving the efficiency of uncertainty propagation in simulation-based multidisciplinary design. Both the parameter uncertainty and model (structure) uncertainty are considered in the proposed procedures. Since the proposed approach is developed for evaluating only the mean and variance of a performance distribution, the first-order sensitivity analysis and the associated linear analysis can be utilized to improve the computational efficiency. Moreover, the approach can be easily integrated with a system optimization process, where the sensitivity information can be shared in between uncertainty propagation and optimization search. The obtained information on the mean and the variance of a system attribute can be used further to assist designers in making reliable design decisions based on techniques such as robust design[16, 17] utility function optimization [18], or other multi-objective optimization techniques[19], and using the state-of-art multidisciplinary optimization frameworks [1, 20, 21]).

Our future work is targeted toward: 1) utilizing the uncertainty propagation methods as a part of robust design optimization for uncertainty mitigation under the MDO framework; 2) developing efficient methods to obtain the complete statistical information of system outputs, i.e., to represent the performance distribution using the probability density functions.

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Appendix – Analytical Relationships in the Electronic Packaging Design

The model of The design variables are:
x_1: The electronic packaging problem:
x_2: Heat sink width (m)
x_3: Heat sink length (m)
x_4: Fin length (m)
x_5: Fin width (m)
x_6: Resistance #1 at temperature T° (Ω)
x_7: Temperature coefficient of electrical resistance #1 (°K^{-1})
x_8: Resistance #2 at temperature #2 (°K^{-1})

The thermal and electrical state variables are:
y_1: negative of watt density (watts/m^3)
y_2: resistance #1 at temperature T_1° (Ω)
y_3: resistance #1 at temperature T_2° (Ω)
y_4: current in resistor #1 (amps)
y_5: current in resistor #2 (amps)
y_6: power dissipation in resistor #1 (watts)
y_7: power dissipation in resistor #2 (watts)
y_8: total circuit current (amps)
y_9: total circuit resistance (Ω)
y_{10}: total current power (watts)
y_{11}: component temperature T_1 of resistor #1 (° C)
y_{12}: component temperature T_2 of resistor #2 (° C)
y_{13}: heat sink volume (m^3)

T_1° (Ω) and T_2° (Ω) are constants which are equal to 20° C.

The following equations describe the above states:

$$y_1 = -y_{10} / y_{13}$$
$$y_2 = x_5[1.0 + x_6(y_{11} - T^c)]$$
$$y_3 = x_7[1.0 + x_8(y_{12} - T^c)]$$
$$y_4 = y_3 y_8 /(y_2 + y_3)$$
$$y_5 = y_2 y_8 /(y_2 + y_3)$$
$$y_6 = y_4^2 y_2$$
$$y_7 = y_5^2 y_3$$
$$y_8 = voltage / y_9$$
$$y_9 = (1.0 / y_2 + 1.0 / y_3)^{-1}$$
$$y_{10} = y_8^2 y_9$$
$$y_{11} = \text{implicity function } (\text{y}_6, y_7, x_1, x_2, x_3, x_4)$$
$$y_{12} = \text{implicity function } (\text{y}_6, y_7, x_1, x_2, x_3, x_4)$$
$$y_{13} = x_1 x_2 x_3$$

where T^c and $voltage$ constants which equal to 10.0 volts.

The system outputs are:
Watt density $f = y_1$
Branch equality current $h = y_4 - y_5$
Component 1 reliability constraint $g_1 = y_{11} - 85$
Component 2 reliability constraint $g_1 = y_{12} - 85$