AIAA 2000-4928

CONCURRENT SUBSYSTEM UNCERTAINTY ANALYSIS IN MULTIDISCIPLINARY DESIGN

Xiaoping Du* and Wei Chen†
Department of Mechanical Engineering
University of Illinois at Chicago
Chicago, Illinois 60607-7022

Abstract

A modified concurrent subsystem uncertainty analysis (MCSSUA) method for uncertainty analysis in multidisciplinary design optimization (MDO) is presented in this paper . Given the probabilistic representations of uncertain parameters and model error estimations, the method quickly evaluates the mean and the variance of a system output so as to improve the computational efficiency of probabilistic optimization such as robust design in the MDO environment. The MCSSUA method is an improved version of the original concurrent subsystem uncertainty analysis (CSSUA) method. The method utilizes the strategy of concurrent subsystem analysis to obtain the mean values of the linking variables via suboptimization. The approximation of a system output is generated at the mean values of input parameters and the mean and variance of the system output are derived. The MCSSUA method is more efficient than the original CSSUA method as the number of design variables in suboptimizations is reduced by half. It is more accurate because no assumption of independence among linking variables is required. A comparison of system uncertainty analysis (SUA) method, the CSSUA method, and the modified CSSUA method is made in the context of highly coupled analyses and multidisciplinary robust design. Two MDO examples are presented to demonstrate the proposed method.

Nomenclature

CSSUA	Concurrent Subsystem Uncertainty Analysis Method
F	simulation model
MCS	Monte Carlo Simulation
MCSSUA	Modified Concurrent Subsystem Uncertainty Analysis Method
n	Number of subsystem (discipline)
SUAM	System Uncertainty Analysis Method
$\mathbf{x}_i$	Input variable of subsystem *i*
$\mathbf{x}_s$	sharing variable
y	linking variable
z	system output
e	model error
m	mean
s	standard deviation

1. Introduction

Uncertainty analysis in multidisciplinary design has been investigated in recent publications[1-9]. Uncertainties exist in every aspect of a design process such as in the model structure and parameters due to the abstractions of the realities, assumptions, lack of knowledge, and random physical properties. Taking uncertainties explicitly into account can help designers make more reliable decisions. There are many methods developed for the purpose of uncertainty analysis, or propagating the effect of uncertainty. Those methods include Monte Carlo simulation[10], first-order second-moment analysis[11, 12], stochastic response surface method[13], and reliability analysis based approaches[14]. However, most of them are applicable only for one discipline or integrated analysis. The challenge is how to efficiently propagate the effect of uncertainty in the context of multidisciplinary analysis and design. Under a multidisciplinary design environment, a system is composed of multidisciplinary subsystems each using a variety of disciplinary models with uncertainties associated with performance predictions. These subsystems are often highly coupled where the performance prediction of one discipline may become the input of another discipline and vice versa. The final

* Visiting Scholar
† Assistant Professor, Member AIAA, weichen1@uic.edu

output from the integrated multidisciplinary system has an accumulated effect of the uncertainties from the individual disciplines. As integrated multidisciplinary design analysis is computationally expensive and the computational demand of individual subsystem varies from one to another, methods need to be developed to bring the features of a MDO system into account.

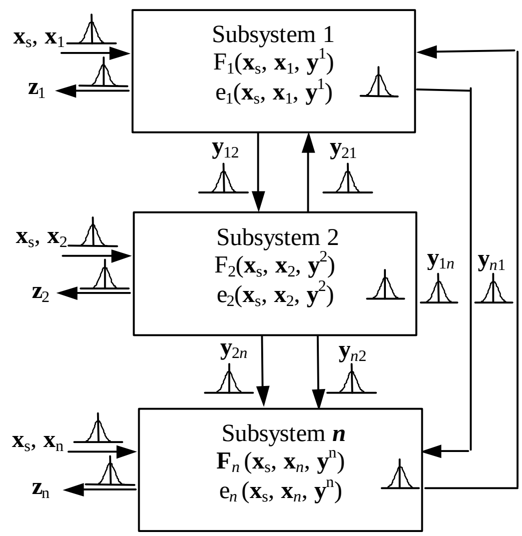
Recent years have seen a few studies for multidisciplinary design with the consideration of model structure uncertainty and parameter uncertainty[2-5]. With Gu's work[2], model uncertainty is denoted by a range (bias) of the system output; the "worst case" concept and the first-order sensitivity analysis are used to evaluate the interval of the end performance of a multidisciplinary system. Du et al.[4] developed efficient uncertainty propagation techniques that could accommodate generic probabilistic representations of uncertain parameters and model error estimations in a multidisciplinary design system (see Section 2 for details). Two techniques, namely, the system uncertainty analysis (SUA) method and the concurrent subsystem uncertainty analysis (CSSUA) method were proposed. The former approach utilizes Taylor approximations as well as local and global sensitivity analysis (first-order derivatives) to evaluate the mean and variance of a system output, while the later uses only local sensitivities and a parallel scheme that allows uncertainty analysis implemented concurrently at the subsystem level. Both techniques have been applied to robust multidisciplinary design[6]. It has been observed that the accuracy of SUA is generally better than that of CSSUA.

In this paper, a modified concurrent subsystem uncertainty analysis (MCSSUA) method is proposed. This method has the same advantage as CSSUA in that the uncertainty analysis is implemented concurrently within each subsystem, but its accuracy is much better because the assumption of independent linking variables is not required any more. A mathematical example and a real engineering example (electronic packaging problem) are used to demonstrate the proposed method. The advantages of the proposed method are demonstrated by comparing it with the SUA method, the original CSSUA method, the MCSSUA method, and the Monte Carlo simulation (MCS).

2. Frame of References

2.1 Problem Formulation

Fig.1 shows the n-discipline system, where each box represents a simulation program that belongs to a discipline (or subsystem) for design evaluation. $\mathbf{x}_s$ are the input variables considered by more than one subsystem, also called sharing variables. $\mathbf{x}_i$ $(\mathrm{i}=1,\mathrm{n})$ are the input variables of subsystem *i*. Note that $\mathbf{x}_s$ and $\mathbf{x}_i$ are mutually exclusive sets of input variables. In general, $\mathbf{x}_s$ and $\mathbf{x}_i$ have uncertainties associated with them which can be expressed by probabilistic distributions.



$F_i=\{\mathbf{F}_{zi}, \mathbf{F}_{yij}, i\neq j\}$, $e_i=\{e_{zi}, e_{yij}, i\neq j\}$

Figure 1 MDO system with

$\mathbf{y}_{ij}(i\neq j)$ are linking variables, which are those functional outputs calculated in subsystem i, at the same time, are required as inputs to subsystem *j*. For simplification of representation, we denote $\mathbf{y}_i=\{\mathbf{y}_{ij}\,|\,j=1,n,\, j\neq i\}$ as the set of linking variables generated as outputs from subsystem *i* and taken as inputs to the other subsystems and $\mathbf{y}^i=\{\mathbf{y}_1,\cdots\mathbf{y}_{i-1},\mathbf{y}_{i+1},\cdots,\mathbf{y}_n\}$ as the set of linking variables generated as outputs from each of the subsystem except subsystem *i* and taken as inputs to subsystem *i*.

For subsystem *i*, based on the subsystem simulation model $\mathbf{F}_{yi}(\cdot)$ and the corresponding model error $\mathbf{e}_{yi}(\cdot)$, the linking variables can be derived as:

$$\mathbf{y}_i=\mathbf{F}_{yi}(\mathbf{x}_s,\mathbf{x}_i,\mathbf{y}^i)+\mathbf{e}_{yi}(\mathbf{x}_s,\mathbf{x}_i,\mathbf{y}^i)\,. \qquad (1)$$

Similarly, the general output of subsystem *i*, $\mathbf{z}_i(\mathrm{i}=1,\mathrm{n})$, can be derived as:

$$\mathbf{z}_i=\mathbf{F}_{zi}(\mathbf{x}_s,\mathbf{x}_i,\mathbf{y}^i)+\mathbf{e}_{zi}(\mathbf{x}_s,\mathbf{x}_i,\mathbf{y}^i)\,. \qquad (2)$$

The outputs of each subsystem $\mathbf{z}_i$, which may include linking variables, are often associated with the system attributes for the evaluations of constraints and objectives in optimization.

In propagating the effect of uncertainty, the goal is to quantify the distributions of system outputs $\mathbf{z}_i$ for the given parameter uncertainty and the model uncertainty. If we choose to use the first and second moments (mean value and variance) to describe the distributions, the problem can be states as:

Given: mean values and variances of input variables
$\boldsymbol{\mu}_{xs}$, $\boldsymbol{\mu}_{xi}$, $\mathbf{s}_{xs}$, and $\mathbf{s}_{xi}$ $(\mathrm{i}=1,\mathrm{n})$
mean values and variances of model errors
$\boldsymbol{\mu}_{yi}$, $\boldsymbol{\mu}_{ezi}$, $\mathbf{s}_{eyi}$ and $\mathbf{s}_{ezi}$ $(\mathrm{i}=1,\mathrm{n})$
Find: mean values and variances of system outputs
$\boldsymbol{\mu}_{zi}$ and $\mathbf{s}_{zi}$ $(\mathrm{i}=1,\mathrm{n})$

Since the MCSSUA method is developed based on the ideas from the system uncertainty analysis (SUA) method and the original concurrent subsystem uncertainty analysis (CSSUA) method, the SUA and the CSSUA are briefly introduced as the following. More details can be found in Ref. 4.

2.2 The SUA method

The SUA is an approach that utilizes Taylor approximations as well as sensitivity analysis (first-order derivatives) to evaluate the variance of a system attribute subject to both parameter and model uncertainties in a multidisciplinary system.

The mean values of linking variables and system outputs are approximated at the mean values of inputs as

$$\boldsymbol{\mu}_{yi} = \mathbf{F}_{yi}(\boldsymbol{\mu}_{xs}, \boldsymbol{\mu}_{xi}, \boldsymbol{\mu}_{y}^{i}) + \boldsymbol{\mu}_{eyi} \quad (3)$$

$$\boldsymbol{\mu}_{zi} = \mathbf{F}_{yi}(\boldsymbol{\mu}_{xs}, \boldsymbol{\mu}_{xi}, \boldsymbol{\mu}_{y}^{i}) + \boldsymbol{\mu}_{ezi} \quad (4)$$

The evaluations of Eqns. (3) and (4) require analyses at the system level.

To obtain the variances of system outputs, first, linking variables $\mathbf{y}_i$ $(\mathrm{i}=1,\mathrm{n})$ are linearized by the first-order Taylor approximations at the system level. Multiple linking variables are derived simultaneously based on a set of linear equations. Second, we approximate a system output by the first-order Taylor expansion with respect to input variables $\mathbf{x}_s$, $\mathbf{x}_i$ and linking variables $\mathbf{y}_i$. After substituting $\mathbf{y}_i$ with the approximation derived earlier, we have the approximation of a system output as the function of input variables $\mathbf{x}_s$, $\mathbf{x}_i$ only. Finally, based on the approximated system output, its variance is evaluated by the following matrix form

$$\mathbf{D}_z = \mathbf{I}\mathbf{D}_{xs} + \mathbf{J}\mathbf{D}_x + \mathbf{K}\mathbf{D}_{ye} + \mathbf{D}_{ze}, \quad (5)$$

where $\mathbf{D}$ stands for vectors of variance and $\mathbf{I}$, $\mathbf{J}$, $\mathbf{K}$ are matrices of derivatives of linking variables and system outputs with respect to input variables (see the appendix for details). From Eqn. (5), it is noted that with this approach the total variation of a system output is derived as the sum of the variations contributed by four individual sources, i.e., the uncertainties of the sharing variables $\mathbf{x}_s$, the variation of subsystem input variables $\mathbf{x}_i$, variation of linking variable $\mathbf{y}_i$ due to model uncertainty, and the variation of system output $\mathbf{z}_i$ due to model uncertainty.

2.3 The CSSUA method

The CSSUA method facilitates the parallelization of the variance evaluation for system outputs. It is developed in such a way that the mean and variance of each of the subsystems can be calculated simultaneously to facilitate the implementation of parallel computing. Here the compatibility among the subsystems (the match of linking variables $\mathbf{y}_i$) is taken care of by matching both the mean values $\boldsymbol{\mu}_{yi}$ and variances $\mathbf{s}_{yi}$. This is achieved by a system level optimizer which sets the target values of the mean and variance of the linking variables and minimizes the deviations between the targets and those that are actually generated through the subsystems analyses. The idea is presented in the following unconstrained optimization model:

Given: mean values and variances of input variables
$\boldsymbol{\mu}_{xs}$, $\boldsymbol{\mu}_{xi}$, $\mathbf{s}_{xs}$, and $\mathbf{s}_{xi}$ $(\mathrm{i}=1,\mathrm{n})$
mean values and variances of error models
$\boldsymbol{\mu}_{yi}$, $\boldsymbol{\mu}_{ezi}$ and $\mathbf{s}_{eyi}$ $(\mathrm{i}=1,\mathrm{n})$
Find: mean values and variances of linking variables
$\boldsymbol{\mu}_{yi}$ and $\mathbf{s}_{yi}$ $(i=1,n)$
Minimize: $d = \min \sum_{i=1}^{n} [(\boldsymbol{\mu}_{yi} - \boldsymbol{\mu}_{yi}^{*})^2 + (\mathbf{s}_{yi} - \mathbf{s}_{yi}^{*})^2]$

$\boldsymbol{\mu}_{yi}^{*}$ and $\mathbf{s}_{yi}^{*}$ (i=1, n) are the mean values and the standard deviations of linking variables $\mathbf{y}_i$. $\boldsymbol{\mu}_{yi}^{*}$ can be calculated by Eqn. (3) and if neglecting the dependence between $\mathbf{y}_i$ and $\mathbf{x}_s, \mathbf{x}_i$, $\mathbf{y}_j\,(j \neq i)$, based on Eqn. (A3), the variance $\mathbf{s}_{yi}^{*}$ of $\mathbf{y}_i$ is obtained as

$$\mathbf{s}_{yi}^{*2} = \sum_{\substack{j=1 \\ j\neq i}}^{n} \left(\frac{\partial \mathbf{F}_{yi}}{\partial \mathbf{y}_j}\right)^2 \mathbf{s}_{yj}^{2} + \left(\frac{\partial \mathbf{F}_{yi}}{\partial \mathbf{x}_s}\right)^2 \mathbf{s}_{xs}^{2} + \left(\frac{\partial \mathbf{F}_{yi}}{\partial \mathbf{x}_i}\right)^2 \mathbf{s}_{xi}^{2} + \mathbf{s}_{eyi}^{2} \quad (6)$$

Once the mean values and variances of $\mathbf{y}_i$ are obtained using the above optimization model, the mean value $\boldsymbol{\mu}_{zi}$ can be determined based on Eqn. (4). If neglecting the dependence between $\mathbf{y}_i$ and $\mathbf{x}_s, \mathbf{x}_i$, based on Eqn. (A4), the variance of $\mathbf{z}_i$ is estimated as:

$$\mathbf{s}_{zi}^{2} = \sum_{j=1}^{n} \left(\frac{\partial \mathbf{F}_{zi}}{\partial \mathbf{y}_j}\right)^2 \mathbf{s}_{yj}^{2} + \left(\frac{\partial \mathbf{F}_{zi}}{\partial \mathbf{x}}\right)^2 \mathbf{s}_{xs}^{2} + \left(\frac{\partial \mathbf{F}_{zi}}{\partial \mathbf{x}_i}\right)^2 \mathbf{s}_{xi}^{2} + \mathbf{s}_{ezi}^{2} \quad (7)$$

The independence assumption is the major cause of inaccuracy associated with the original CSSUA.

3. Modified Concurrent Subsystem Uncertainty Analysis

3.1 MCSSUA

From the introduction of SUA and CSSUA in Section 2, we note that for the evaluation of the mean value of a system output, only one analysis at the system level is required with the SUA method. No analysis at the system level is required with the CSSUA method. For the calculation of variance, the SUA solves the simultaneous linear equations to obtain the linear approximations of linking variables $\mathbf{y}$. Therefore, the total number of system level analysis is one with the SUA method. For the CSSUA, no system level analysis is needed for the mean and variance evaluations. Hence CSSUA is more suitable for the situation under which the system level analysis is very expensive. Even though no system level analysis is required, the efficiency and accuracy of the original CSSUA still need to be improved. Rooms of improvement are examined as the following.

From the aspect of efficiency, first, the original CSSUA requires lots of subsystem analyses which are used for three purposes: 1) evaluating the mean values of linking variable in the suboptimization, 2) evaluating variances of linking variables in the suboptimization and 3) evaluating variances of system outputs after the suboptimization. In each function call of the suboptimiztion, most of the subsystem analyses are needed for purpose 2) since the calculation of variance involves the evaluations of derivatives, of which the number of evaluations is proportional to the number of variables in the function of a linking variable. Meanwhile only one subsystem analysis is required for evaluating the mean values of linking variables. Second, the number of unknown variables in suboptimization of the original CSSUA method is quite large, which is twice as large as the total number of linking variables in the system.

From the aspect of accuracy, the original CSSUA assumes that linking variables are independent with each other. Since this assumption barely holds in most cases, the accuracy of the CSSUA is generally not as good as the SUA method.

To improve the efficiency and accuracy of the original CSSUA method while keeping the feature of simultaneous (concurrent) subsystem analyses, we propose a modified CSSUA method, which does not need any subsystem analysis for purpose 2) and the independent assumption for linking variables. The number of unknown variables in suboptimization is reduced by half to be exactly equal to the number of linking variables. The procedure is as follows.

1) Find mean values of linking variable by the suboptimization as shown in Fig. 2.

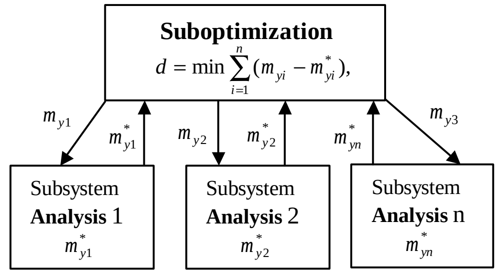


Figure 2 Suboptimization in MCSSUA

Here, the compatibility of the system is achieved by an optimizer which sets the target values of the mean values of linking variables and minimizes the deviations between the targets and those that are actually generated through the subsystems analyses. The idea can be generated as the following unconstrained optimization model:

Given:	mean values of input variable $\boldsymbol{\mu}_{xs}$ and $\boldsymbol{\mu}_{xi}$ $(\mathrm{i} = 1, \mathrm{n})$
Find:	mean values of linking variable $\boldsymbol{\mu}_{yi}$ $(\mathrm{i} = 1, \mathrm{n})$
Minimize:	$d = \min \sum_{i=1}^{n} (\boldsymbol{\mu}_{yi} - \boldsymbol{\mu}_{yi}^*)^2$

$\boldsymbol{\mu}_{yi}$ are the unknown variables in the suboptimization and $\boldsymbol{\mu}_{yi}^*$ are the mean values of linking variables evaluated in subsystems.

2) Evaluate the mean value of a system output

The mean value of a system output is evaluated by substituting the mean of linking variable $\boldsymbol{\mu}_{yi}$ in Eqn. (3) with the suboptimization result.

3) Evaluate variance of a system output

Following the same principle of the SUA, we obtain the system variance as in Eqn. (5) (see the appendix for details).

From the above procedure, we see that the principle of the MCSSUA is to first obtain the mean values of the linking variables by suboptimization which only utilizes subsystem analyses. The approximation of a system output is then obtained at the mean values of system input and linking variables Mean and variance are then derived based on the approximated function. Based on its working principle, the MCSSUA is expected to result in the following improvements: 1) The number of unknown variables in the suboptimization is only half of the number of unknown variables in the suboptimization of CSSUA. The reduced problem size will be helpful to the convergence of the suboptimization. 2) The computational effort decreases because the subsystem analysis for purpose 2) (evaluating derivatives for variance calculations) is no longer needed. 3) The accuracy of MCSSUA for variance evaluation is better than that of the CSSUA without the assumption of independence of linking variables. If the suboptimizatiom generates same mean values of linking variables as the SUA, the MCSSUA will have the same result as the SUA. 4) The MCSSUA can still be implemented under the parallel computing environment using concurrent subsystem analyses.

3.2 Comparison of the different methods

Before we demonstrate the comparative performance of various methods for uncertainty analysis through our examples, we first discuss in principle the comparative efficiency and accuracy of the SUA, the CSSUA and the MCSSUA methods.

Efficiency

Efficiency is measured by the number of system and subsystem analyses. The SUA method needs one system level analysis while the CSSUA and MCSSUA methods do not require any system level analysis. Hence, if the system level analysis is very expensive, we may consider adopt the CSSUA and MCSSUA for analysis and design when the computational burden is concerned. On the other hand, the CSSUA and MCSSUA method need more subsystem level analyses (subsystem analyses) than the SUA due to the suboptimization. If the derivatives are evaluated numerically, the number of subsystem analysis for each different method is summarized as the following.

1) The number of subsystem analysis for SUA

$$N_{SUA} = \sum_{i=1}^{n} [N_{y\_output}(i) + N_z(i)] \times [1 + N_{xs} + N_x(i) + N_{y\_input}(i)] \tag{8}$$

where

$N_{y\_output}(i)$ - number of linking variable $\mathbf{y}_i$ (the output for subsystem i),

$N_z(i)$ - number of system output for subsystem i,

N_{xs} - number of sharing input variables,

$N_x(i)$ - number of input variables for subsystem i,

$N_{y\_input}(i)$ - number of linking variables $\mathbf{y}^i$ (the input for subsystem i).

2) The number of subsystem analysis for the CSSUA

$$N_{CSSUA} = [N_{function\_call} \sum_{i=1}^{n} N_{y\_output}(i) + \sum_{i=1}^{n} N_z] [1 + N_{xs} + N_x(i) + N_{y\_input}(i)] \tag{9}$$

where $N_{function\_call}$ is the number of function evaluations in the suboptimization, which depends on the optimization algorithm, starting point, convergence criteria, etc.

3) The number of subsystem analysis for the MCSSUA

$$N_{MCSSUA} = N_{function\_call} \sum_{i=1}^{n} N_{y\_output}(i) + \sum_{i=1}^{n} N_z [1 + N_{xs} + N_x(i) + N_{y\_input}(i)] \tag{10}$$

It is seen that the number of subsystem analyses of the original CSSUA method is approximately $N_{function\_call}$ times as large as the SUA and $N_{function\_call} \sum_{i=1}^{n} N_{y\_output}(i)[N_{xs} + N_x(i) + N_{y\_input}(i)]$ more than the MCSSUA. The difference of the number of subsystem analysis of the SUA and the MCSSUA is the number of function calls (in suboptimization) times the total number of linking variables. To reduce the number of subsystem analyses in MCSSUA, choosing suitable optimization algorithm, starting points and other settings for optimization is very important.

Accuracy

All the three methods, the SUA, CSSUA and the MCSSUA are fundamentally approximation methods for uncertainty analysis. The first- and second- order moments (mean and variable) are generally not sufficient enough to capture the full probabilistic distribution of a system output, although they are quite sufficient for some applications such as robust design. Another source of approximation is the Taylor expansion. All the linking variables and system outputs are expanded by the first-order serial at means of input variables and input linking variables. When the variances of input variables are big or the functions are highly nonlinear, the error by the Taylor expansion will become large. For the CSSUA, since it is assumed that all the linking variables are independent with each other, a condition barely holds in reality, its accuracy is generally not as good as that of the SUA and the CSSUA. However, this conclusion may not be always true since the independence assumption is not the unique error source and different error sources may compensate with each. In principle, if the suboptimization in MCSSUA generate same linking variables as the SUA, theoretically, the results from MCSSUA will be the same as those from the SUA. Due to the numerical difficulties for optimization, accurate estimations of linking variables through suboptimization are not guaranteed. Hence, practically, it is possible that the SUA and MCSSUA may generate different results.

3.3 Integration of the Methods for Uncertainty Analysis within the MDO Infrastructures

When uncertainties are considered in MDO, the SUA, the CSSUA and the MCSSUA could become useful techniques for uncertainty analysis under the general framework of multidisciplinary design optimization (see Fig. 3).

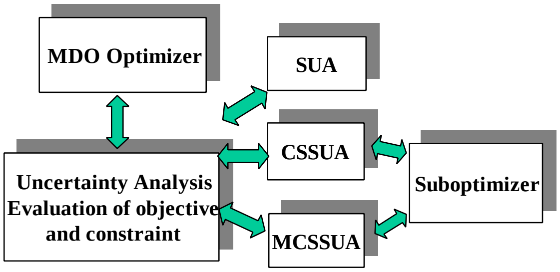


Figure 3 MDO with uncertainty analysis

When choosing the appropriate uncertainty analysis technique in MDO, we need to consider not only the efficiency and the accuracy but also the compatibility between the uncertainty analysis method and the MDO infrastructure intended to use. In general, the SUA can work with a single-level MDO infrastructure level (for example, all-in-one MDO approach), while the CSSUA and the MCSSUA can work with the multilevel MDO approaches (for example, the CSSO MDO approach) [17]. Here, the suboptimizer for uncertainty analysis are placed at the sublevel of MDO.

4. Examples

Our proposed method is demonstrated through multidisciplinary robust design[19, 20] applications where the mean and the variance of an objective are minimized simultaneously subject to robust feasibility[12]. To compare the merits of various approaches, we investigate the accuracy of using the uncertainty analysis techniques for both multidisciplinary design evaluations (analyses) and optimization. In each comparison, the result from the large amount (10^6 simulation number) of Monte Carlo Simulations (MCS) is considered as the correct solution.

4.1 A Mathematical Example

Problem statement

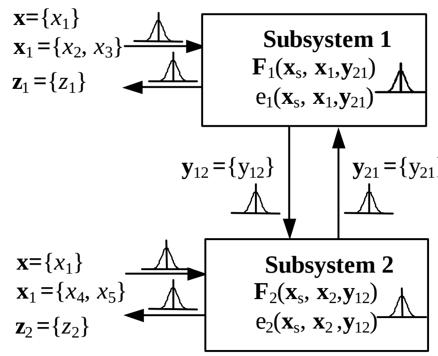
Two subsystems are considered (Fig. 4) and the functional relationships are represented as:

Subsystem1

$$\mathbf{x}_s = \{x_1\}, \quad \mathbf{x}_1 = \{x_2, x_3\}, \quad \mathbf{y}_1 = \mathbf{y}_{12} = \{y_{12}\},$$
$$\mathbf{z}_1 = \{z_1\}, \; \mathbf{e}_{y1}(\mathbf{x}_s, \mathbf{x}_1, \mathbf{y}_1) = \{e_{y12}(\mathbf{x}_s, \mathbf{x}_1, \mathbf{y}_2)\},$$
$$\mathbf{e}_{z1}(\mathbf{x}_s, \mathbf{x}_1, \mathbf{y}_1) = \{e_{z1}(\mathbf{x}_s, \mathbf{x}_1, \mathbf{y}_1)\},$$
$$\mathbf{F}_{y12}(\mathbf{x}_s, \mathbf{x}_1, \mathbf{y}_1) = \{F_{y12}(\mathbf{x}_s, \mathbf{x}_1, \mathbf{y}_1)\}$$
$$= x_1^2 + 2x_2 - x_3 + 2\sqrt{y_{21}}$$
$$\mathbf{F}_{z1}(\mathbf{x}_s, \mathbf{x}_1, \mathbf{y}_1) = \{F_{z1}(\mathbf{x}_s, \mathbf{x}_1, \mathbf{y}_1)\}$$
$$= x_1^2 + 2x_2 + x_3 + x_2 e^{-y_{21}}$$

Subsystem 2

$$\mathbf{x}_s = \{x_1\}, \; \mathbf{x}_2 = \{x_4, x_5\}, \; \mathbf{y}_2 = \mathbf{y}_{21} = \{y_{21}\},$$
$$\mathbf{z}_2 = \{z_2\}, \; \mathbf{e}_{y2}(\mathbf{x}_s, \mathbf{x}_2, \mathbf{y}_2) = \{e_{y21}(\mathbf{x}_s, \mathbf{x}_2, \mathbf{y}_1)\},$$
$$\mathbf{e}_{z2}(\mathbf{x}_s, \mathbf{x}_2, \mathbf{y}_2) = \{e_{z2}(\mathbf{x}_s, \mathbf{x}_2, \mathbf{y}_2)\},$$
$$\mathbf{F}_{y21}(\mathbf{x}_s, \mathbf{x}_2, \mathbf{y}_2) = \{F_{y21}(\mathbf{x}_s, \mathbf{x}_2, \mathbf{y}_2)\}$$
$$= x_1 x_4 + x_4^2 + x_5 + y_{12}$$
$$\mathbf{F}_{z2}(\mathbf{x}_s, \mathbf{x}_2, \mathbf{y}_2) = \{F_{z2}\} = \sqrt{x_1} + x_4 + x_5(0.4 y_{12})$$



$\mathbf{F}_1=\{\mathbf{F}_{z1}, \mathbf{F}_{y12}\}$, $\mathbf{F}_2=\{\mathbf{F}_{z2},\mathbf{F}_{y21}\}$

Figure 4 Information Flow– Example

The parameter and model error uncertainties are represented by distributions as defined in Table 1.

Table 1. Parameter and Model Uncertainties

	m	s	Distribution
x_1	m_{x1}	$0.1m_{x1}$	Normal
x_2	m_{x2}	$0.1m_{x2}$	Normal
x_3	m_{x3}	$0.1m_{x3}$	Normal
x_4	m_{x4}	$0.1m_{x4}$	Normal
x_5	m_{x5}	$0.1m_{x5}$	Normal
e_{y12}	0	$0.1m_{y12}$	Normal
e_{y21}	0	$0.1m_{y21}$	Normal
e_{z1}	0	$0.1m_{z1}$	Normal
e_{z2}	0	$0.1m_{z2}$	Normal

Design Analysis

Means and standard deviations of the system outputs z_1 and z_2 are calculated by the SUA, the CSSUA, and the MCSSUA at three design points $(m_{x1}, m_{x2}, m_{x3}, m_{x4}, m_{x5}) = (1, 1, 1, 1, 1)$, (2, 2, 2, 2, 2) and (2, 5, 2, 5, 2). The results are compared in Figs 5, 6, and 7.

From the figures, it is seen that at all the three points, means of z_1 and z_2 from the SUA, the CSSUA and the MCSSUA methods are almost identical to the results from the MCS. The standard deviations of z_1 and z_2 from all the methods are very close with each other. The results from the SUA and the MCSSUA are exactly the same which means the suboptimization in the MCSSUA generates the same values of the linking variables as the SUA does by solving the set of simultaneous equations. The last columns of the tables (for standard deviation of z_2) in all three figures show that the SUA and the MCSSUA are more accurate than the CSSUA. This matches with the discussion in Section 3.2 which states that the SUA and the MCSSUA are expected to have better accuracy for variance evaluation than the CSSUA.

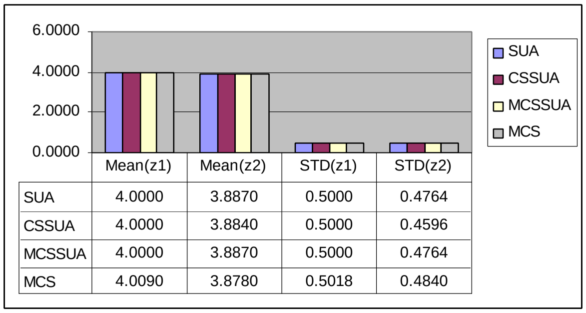


Figure 5. Analysis result at design point 1

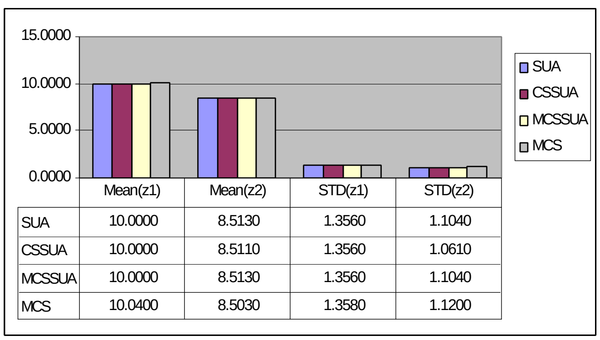


Figure 6. Analysis result at design point 2

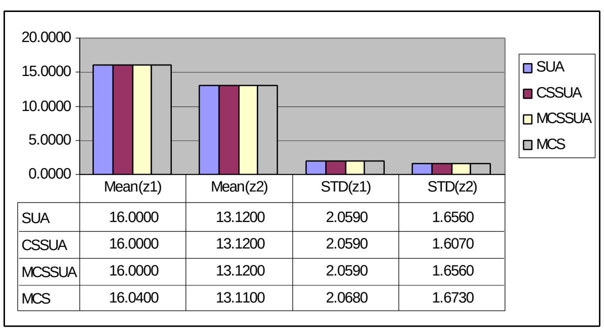


Figure 7. Analysis result at design point 3

When applying the CSSUA for this mathematical problem, there are four unknown variables (two for the means of the two linking variables and two for their variances). With the MCSSUA, this number is reduced to two (for the means of the two linking variables) in suboptimization.

Optimization

For this example, the optimization model without uncertainty is represented as:

Find: the values of $x_1 \sim x_5$

Minimize: z_2

Subject to: $11 - z_1 \le 0$

$12 - z_2 \le 0$

When considering uncertainty, the optimization model is converted to a robust design model as:

Find: the mean values $m_{x1} \sim m_{x5}$

Minimize: $w_1 \dfrac{m_{z2}}{m^*_{z2}} + w_2 \dfrac{s_{z2}}{s^*_{z2}}$ (objective)

Subject to: $11 - (m_{z1} + k s_{z1}) \le 0$ (constraint 1)

$12 - (m_{z2} + k s_{z2}) \le 0$ (constraint 2)

The w_1 and w_2 in the above model are the weighting factors with $w_1 + w_2 = 1$. k is chosen to be 1 which indicates that with 84.13% probability, the constraint will be satisfied under the assumption that constraint functions are normally distributed. m^*_f (obtained by $w_1 = 1$ and $w_2 = 0$) and s^{*2}_f (obtained by $w_1 = 0$ and $w_2 = 1$) are the ideal solutions used to normalize the two aspects in the objective, i.e., optimizing the mean performance and minimizing performance deviations. Table 2 lists the robust design solutions for this multidisciplinary system from using all the four methods.

Table 2 Robust optimization results of example 1

	SUA	CSSUA	MCSSUA	MCS
m_{x1}	2.0700	2.0706	2.0532	2.1106
m_{x2}	0.7724	0.9695	0.9963	1.0104
m_{x3}	1.8234	1.4256	1.4822	1.4885
m_{x4}	0.0001	0.0	0.0043	0.1160
m_{x5}	3.5550	3.4818	3.4347	3.3227
Objective	4.1859	4.2549	4.1836	4.1697
Constraint 1	-0.0445	-0.0445	-0.0698	-0.4517
Constraint 2	-0.0423	-0.2273	-0.0369	0.0

The values of the objective and constraints in the last three rows for the SUA, the CSSUA and the MCSSUA are the results confirmed by the MCS based on the optimal solutions of m_{x1} to m_{x5}. Although the solutions of m_{x1} to m_{x5} slightly vary from one method to another, we note that the SUA, the CSSUA, and the MCSSUA methods all generate very close optimal solution to that from the MCS, in terms of the value of the objective function and the feasibility. We also find that the values of the objective function from the SUA and the MCSSUA methods are closer to MCS compared to the one from CSSUA.

4.2 Electronic Packaging Problem

Problem statement

The electronic packaging problem is a benchmark multidisciplinary problem comprising the coupling between electronic and thermal subsystems. Component resistances (in electronic subsystem) are affected by operating temperatures in (thermal subsystem), while the temperatures depend on the resistances. The subsystem relationship is demonstrated in Figure 8. A detailed problem statement is provided in Ref. 4 and at http://fmad-www.larc.nasa.gov/mdob/MDOB.

The system analysis consists of the coupled thermal and electrical analyses. The component temperatures calculated in the thermal analysis are needed in the electrical analysis in order to compute the power dissipation of each resistor. Likewise, the power dissipation of each component must be known in order for the thermal analysis to compute the temperatures.

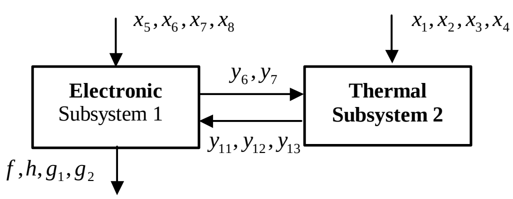


Figure 8 Electrical Package Problem

There are eight input variables x_1~ x_8, five linking variables y_6, y_7, y_{11}, y_{12}, y_{13}, and four system outputs f, h, g_1 and g_2, where f stands for objective, h and g stands for constraint. The sets of variables and functions in the two subsystems are as follows:

Electronic Analysis:

Input variables:

$\mathbf{x}_s = \{f\}$, $\mathbf{x}_1 = \{x_5, x_6, x_7, x_8\}$

Linking variables:

$\mathbf{y}_{21} = \{y_6, y_7\}$

System outputs:

$\mathbf{z}_1 = \{f, h, g_1, g_2\}$

$\{f\}$ stands for an empty set.

Thermal Analysis:

Input variables:

$\mathbf{x}_s = \{f\}$, $\mathbf{x}_2 = \{x_1, x_2, x_3, x_4\}$

Linking variables:

$\mathbf{y}_{12} = \{y_{11}, y_{12}, y_{13}\}$

System outputs:

$\mathbf{z}_2 = \{f\}$

Of the two subsystems, the thermal analysis is more complex, which requires a finite difference solution for the temperature distribution calculation. The remaining equations in the thermal subsystem are solved algebraically. All equations of the electrical system are solved algebraically.

The original electronic packaging problem involves only deterministic analyses where no uncertainty is considered. To illustrate the proposed uncertainty analysis method, we assume uncertainties are associated with the input variables x_i (i = 1,2,…, 8) and the thermal simulation model, both described by normal distributions. The variation coefficient (the ratio of the standard deviation over the mean) of x_i is 0.1. The variation coefficients of the model errors for linking variables y_{11} and y_{11} are also 0.1, i.e., y_{11}~N(m_{y11} ,5.0), y_{12} ~N(m_{y12} ,5.0).

Design Analysis

The accuracies of the SUA, the CSSUA, and the MCSSUA methods for design evaluations are first compared at two design points with the results from the MSC. The two points are:

Point 1: x_1~N(0.1, 0.01), x_2~N(0.1, 0.01), x_3~N(0.1, 0.001), x_4~N(0.05, 0.005), x_5~N(100, 10), x_6~N(0.004, 0.0004), x_7~N(100, 10), x_8~N(0.004, 0.00041).

Point 2: x_1~N(0.08, 0.008), x_2~N(0.08, 0.008), x_3~N(0.055, 0.0055), x_4~N(0.0275, 0.00275), x_5~N(505, 50.5, x_6~N(0.0065, 0.00065, x_7~N(505, 50.5), x_8~N(0.0065, 0.00065).

The results are shown in Tables 3 and 4.

Table 3 Means and standard deviations of system output at point 1

Method	m_f	s_f	m_h	s_f	m_{g1}	s_f	m_{g2}	s_f
SUA	-1.8470E+03	5.3630E+03	3.9010E-06	1.3400E-02	-4.4110E+01	4.0460E+00	-4.4100E+01	4.3900E+00
CSSUA	-1.8480E+03	5.2920E+03	5.3440E-06	1.3220E-02	-4.4110E+01	4.0890E+00	-4.4100E+01	4.0900E+00
MCSSUA	-1.8470E+03	5.3630E+03	3.9010E-06	1.3400E-02	-4.4110E+01	4.0460E+00	-4.4100E+01	4.3900E+00
MCS	-1.8290E+03	5.3190E+03	1.0000E-11	1.3340E-02	-4.4030E+01	4.1370E+00	-4.4060E+01	4.1300E+00

Table4 Means and standard deviations of system output at point 2

Method	m_f	s_f	m_h	s_f	m_{g1}	s_f	m_{g2}	s_f
SUA	-1.0180E+03	1.0490E+03	1.7130E-07	2.5940E-03	-4.8870E+01	3.5860E+00	-4.8870E+01	3.6490E+00
CSSUA	-1.0190E+03	1.0430E+03	4.9500E-07	2.6000E-03	-4.8870E+01	3.6130E+00	-4.8870E+01	3.6130E+00
MCSSUA	-1.0180E+03	1.0490E+03	1.7130E-07	2.5940E-03	-4.8870E+01	3.5860E+00	-4.8870E+01	3.6490E+00
MCS	-1.0250E+03	1.0690E+03	-2.4100E-10	2.6630E-03	-4.8850E+01	3.6090E+00	-4.8880E+01	3.6070E+00

From Tables 3 and 4, it is noted that the mean values generated by the SUA, the CSSUA and the MCSSUA are very close. Those results under h are small enough to be considered all as zeros.

The estimations of standard deviations by using the SUA, the CSSUA, and the MCSSUA are considered to be satisfactory, while the accuracies of the SUA and the MCSSUA are close and generally better than the CSSUA. However, the CSSUA is less accurate than the SUA and the MCSSUA.

Optimization

The original deterministic optimization model of the electronic packaging problem is represented as:

Find: $x_1 \sim x_8$

Minimize: f

Subject to: $h \le 0$

$g_1 \le 0$

$g_2 \le 0$

When the uncertainty is considered, robust design optimization is formulated as

Find: the mean values $m_{x1} \sim m_{x8}$

$$\text{Minimize: } w_1 \frac{m_f}{m_f^*} + w_2 \frac{s_f}{s_f^*} \quad \text{(objective)}$$

Subject to: $m_h + k s_h \le 0$ (constraint 1)

$m_{g1} + k s_{g1} \le 0$ (constraint 2)

$m_{g2} + k s_{g2} \le 0$ (constraint 3)

The optimum solutions by different methods are listed in Table 5.

Table 5. Optimum results of example 2

	SUA	CSSUA	MCSSUA	MCS
m_{x1}	0.1487	0.050	0.1402	0.1453
m_{x2}	0.0634	0.050	0.0578	0.0641
m_{x3}	0.0187	0.010	0.0155	0.0148
m_{x4}	0.0387	0.0387	0.0336	0.0386
m_{x5}	1000.0	1000.0	1000.0	1000.0
m_{x6}	0.0090	0.0053	0.0090	0.0090
m_{x7}	870.6685	876.9904	875.1554	871.5836
m_{x8}	0.0089	0.0056	0.0075	0.0089
Objective	0.3632	0.7505	0.3656	0.3539
Constraint 1	0.0001	0.0006	-0.0001	-0.0001
Constraint 2	-45.7486	-45.8805	-45.7289	-45.7576
Constraint 3	-45.6685	-45.8805	-45.6442	-45.7057

Similar to the mathematical example presented earlier, the values of the objective and constraints for the SUA, the CSSUA and the MCSSUA in Table 5 are the results confirmed by the MCS based on the optimal solutions identified by these different techniques.

We note that the SUA and the MCSSUA generate optimum solutions that are close to those from the MCS, both in the design variable space and the objective space. The most accurate method is the SUA The results of the SUA and the MCSSUA are slightly different. The accuracy of the CSSUA is the worst because the robust design solution identified is far away from the rest of solutions and the minimized objective function value is the worst. The results from all the techniques tested are feasible.

5. Closure

A modified concurrent subsystem uncertainty analysis (MCSSUA) method for propagating the effect of uncertainty in the multidisciplinary design optimization (MDO) is presented in this paper. The method improves the original concurrent uncertainty analysis (CSSUA) method from two aspects: 1) The efficiency is improved by decreasing the computational effort for subsystem analyses and reducing the number of design variables in the suboptimization and 2) The accuracy is improved by avoiding the assumption of independent linking variables. The comparison of the system analysis (SUA) method, the CSSUA and the MCSSUA is made from both the theoretical point and applications to example problems. We note that depending on the number of variables and the number of disciplines involved, the effectiveness of these techniques varies. The examples illustrate that all the three uncertainty analysis methods are applicable in MDO applications within reasonable tolerance range. Since MCSSUA is generally more efficient and accurate than the CSSUA, the final selection will be between MCSSUA and SUA, depending on the number of disciplines involved, the amount of linking variables, the computational needs of subsystem analysis and integrated system analysis. It should be noted that all the three methods investigated are approximation methods since the Taylor expansion and moment marching methods are employed. If high accuracy is needed (for example in reliability analysis and design) or the full distribution of a system output is needed, other approaches may be better suited. However, these approaches are expected to be more much computationally expensive.

Acknowledgement

The support from the NSF/DMII 9896300 is gratefully acknowledged.

References

1. Sues, R. H., Rhodes, G. S. and Oakley, D. R., "Multidisciplinary Stochastic Optimization," *Proceeding of the 10th Conference on Engineering mechanics*, May, 1995, Boulder, CO, pp. 934-937.
2. Gu, X, Renaud, J. E., and Batill, S. M., "An Investigation of Multidisciplinary Design Subject to Uncertainties," *AIAA 98-4747, 7th AIAA/USAF/NASA/ISSMO Multidisciplinary Analysis & Optimization Symposium*, St. Louise, Missouri, 1998, pp. 309-319.
3. Du, X. and Chen, W., "Propagation and Management of Uncertainties in Simulation-Based Collaborative Systems Design," *The 3rd World Congress of structural and Multidisciplinary Optimization*, May 17-21, 1999, Buffalo, New York.
4. Du, X. and Chen, W., "An Efficient Approach to Probabilistic Uncertainty Analysis in Simulation-Based Multidisciplinary Design," *the 38th AIAA Aerospace Sciences Meeting and Exhibit*, 2000-0423, AIAA, Washington D.C., 2000
5. Kowal, M. and Mahadevan S., 1998, "Uncertainty-Based Multidisciplinary Design Optimization," *7TH AIAA/USAF/NASA/ISSMO Symposium on Multidisciplinary Analysis and Optimization*, St. Lois, MO.
6. Du, X. Wang Y. and Chen, W., "Method for Robust Multidisciplinary Design," *the 41st AIAA /ASME /ASCE/ AHS/ ASC Structures, Structural Dynamics, and Materials Conference*, AIAA, Washington D.C., 2000
7. DeLaurentis D. and Marvris D., 2000, "Uncertainty Modeling and Management in Multidisciplinary Analysis and Synthesis," *the 38th AIAA Aerospace Sciences Meeting*, January 10-13, 2000, Reno Hilton, Reno, NV.
8. Koch, P.N., Simpson, T.W., Allen, J.K., and Mistree F., 1999, "Statistical Approximations for Multidisciplinary Design Optimization: The Problem of Size," *Journal of Aircraft*, Vol. 36, No. 1, pp. 275-286.
9. Mavris, D.V, Bandte, O. and DeLaurentis, D.A., 1999, "Robust Design Simulation: A Probabilistic Approach to Multidisciplinary Design," *Journal of Aircraft*, Vol. 36, No. 1, pp. 298-397.
10. Law, A. M., and Kelton, W. D., *Simulation Modeling and Analysis*, McGraw-Hill Company, 1982, New York.
11. Hoybye, Jan A., "Model error propagation and data collection design, An application in water quality modeling," *Water, Air and Soil Pollution*, Vol. 103, n. 1-4, 1998, pp. 101-119.
12. Du, X. and Chen, W., "Towards a Better Understanding of Modeling Feasibility Robustness in Engineering Design," 1999 ASME Design Technical Conference, Paper No. DAC-8565, Las Vegas, Nevada, Sept. 1999.
13. Isukapalli, S.S, and Georgopoulos, P.G., "Stochastic Response Surface Methods (SRSMs) for uncertainty propagation: Application to environmental and biological systems," *Risk Analysis*, Vol. 18, 1998, pp. 351-363.
14. Wu, Y.-T., Millwater, H.R., and Cruse T. A., "An Advance Probabilistic Analysis Method for Implicit Performance Function," AIAA Journal, 1990, Vol. 28, pp. 1663-1669.
15. Wang Cheng,, parametric Uncertainty Analysis for Complex Engineering Systems, Ph.D. Thesis, June, 1999, MIT.
16. Balling, R. J. and Sobieski, J., "An Algorithm for Solving the System-Level Problem in Multilevel Optimization," *5th AIAA/USAF/NASA/ISSMO Symposium on Recent Advances in Multidisciplinary Analysis and Optimization*, Panama City, Florida, 1994, pp. 794-809.
17. Renaud, J. E., and Tappeta, R. V., "Multiobjective Collaborative Optimization," *ASME Journal of Mechanical Design*, Vol. 119, No. 3, 1997, pp. 403-411.
18. Rao, S.S., 1992, *Reliability-Based Design*, McGraw-Hill, Inc.
19. Chen, W., Allen, J. K., Tsui, K-L, and Mistree, F., "A Procedure for Robust Design," *Transaction of the ASME Journal, Journal of Mechanical Design*, Vol. 118, 1996, pp. 478-485.
20. Parkinson, A., Sorensen, C. and Pourhassan, N., "A General Approach for Robust Optimal Design," *Transactions of the ASME*, Vol. 115, 1993, pp.74-80.

Appendix: Derivation of Variance in the SUA and the MCSSUA

From Eqn. (2), the linking variables $\mathbf{y}_i$ are approximated using Taylor's expansion as

$$\Delta\mathbf{y}_i = \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{\partial \mathbf{F}_{yi}}{\partial \mathbf{y}_j}\Delta\mathbf{y}_j + \frac{\partial \mathbf{F}_{yi}}{\partial \mathbf{x}_s}\Delta\mathbf{x}_s + \frac{\partial \mathbf{F}_{yi}}{\partial \mathbf{x}_i}\Delta\mathbf{x}_i + \Delta\mathbf{e}_{yi}$$

(i=1, n), (A1)

which yields

$$\mathbf{A}\Delta\mathbf{y} = \mathbf{B}\Delta\mathbf{x}_s + \mathbf{C}\Delta\mathbf{x} + \mathbf{D}, \tag{A2}$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{I}_1 & -\frac{\partial \mathbf{F}_{y1}}{\partial \mathbf{y}_2} & \cdots & -\frac{\partial \mathbf{F}_{y1}}{\partial \mathbf{y}_n} \\ -\frac{\partial \mathbf{F}_{y2}}{\partial \mathbf{y}_1} & \mathbf{I}_2 & \cdots & -\frac{\partial \mathbf{F}_{y2}}{\partial \mathbf{y}_n} \\ \cdots & \cdots & \cdots & \cdots \\ -\frac{\partial \mathbf{F}_{yn}}{\partial \mathbf{y}_1} & -\frac{\partial \mathbf{F}_{yn}}{\partial \mathbf{y}_2} & \cdots & \mathbf{I}_n \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \frac{\partial \mathbf{F}_{y1}}{\partial \mathbf{x}_s} \\ \frac{\partial \mathbf{F}_{y2}}{\partial \mathbf{x}_s} \\ \cdots \\ \frac{\partial \mathbf{F}_{yn}}{\partial \mathbf{x}_s} \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} \frac{\partial \mathbf{F}_{y1}}{\partial \mathbf{x}_1} & 0 & \cdots & 0 \\ 0 & \frac{\partial \mathbf{F}_{y2}}{\partial \mathbf{x}_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \frac{\partial \mathbf{F}_{yn}}{\partial \mathbf{x}_n} \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} \mathbf{e}_{y1} - \boldsymbol{\mu}_{ey1} \\ \mathbf{e}_{y2} - \boldsymbol{\mu}_{ey2} \\ \cdots \\ \mathbf{e}_{yn} - \boldsymbol{\mu}_{eyn} \end{bmatrix}$$

$$\Delta\mathbf{x}_s = \mathbf{x}_s - \bar{\mathbf{x}}_s, \quad \Delta\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 - \boldsymbol{\mu}_{x1} \\ \mathbf{x}_2 - \boldsymbol{\mu}_{x2} \\ \cdots \\ \mathbf{x}_n - \boldsymbol{\mu}_{xn} \end{bmatrix}, \text{ and } \Delta\mathbf{y} = \begin{bmatrix} \mathbf{y}_1 - \boldsymbol{\mu}_{y1} \\ \mathbf{y}_2 - \boldsymbol{\mu}_{y2} \\ \cdots \\ \mathbf{y}_n - \boldsymbol{\mu}_{yn} \end{bmatrix}.$$

$\mathbf{I}_i$ (i = 1, n) are the identity matrixes.

Solving Eqn. (A2), we have

$$\Delta\mathbf{y} = \mathbf{A}^{-1}\mathbf{B}\Delta\mathbf{x}_s + \mathbf{A}^{-1}\mathbf{C}\Delta\mathbf{x} + \mathbf{A}^{-1}\mathbf{D}. \tag{A3}$$

System outputs are approximated as

$$\Delta\mathbf{z}_i = \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{\partial \mathbf{F}_{zi}}{\partial \mathbf{y}_j}\Delta\mathbf{y}_j + \frac{\partial \mathbf{F}_{zi}}{\partial \mathbf{x}_s}\Delta\mathbf{x}_s + \frac{\partial \mathbf{F}_{zi}}{\partial \mathbf{x}_i}\Delta\mathbf{x}_i + \Delta\mathbf{e}_{zi}$$

(i=1, n). (A4)

The reorganization of the above equation yields

$$\begin{aligned} \Delta\mathbf{z} &= \mathbf{E}?\,\mathbf{y} + \mathbf{F}\Delta\mathbf{x}_s + \mathbf{G}\Delta\mathbf{x} + \mathbf{H} \\ &= [\mathbf{E}(\mathbf{A}^{-1}\mathbf{B}) + \mathbf{F}]\Delta\mathbf{x}_s + [\mathbf{E}(\mathbf{A}^{-1}\mathbf{C}) + \mathbf{G}]\Delta\mathbf{x} \\ &\quad + \mathbf{E}\mathbf{A}^{-1}\mathbf{D} + \mathbf{H}, \end{aligned} \tag{A5}$$

where

$$\mathbf{E} = \begin{bmatrix} 0 & \frac{\partial \mathbf{F}_{z1}}{\partial \mathbf{y}_2} & \cdots & \frac{\partial \mathbf{F}_{z1}}{\partial \mathbf{y}_n} \\ \frac{\partial \mathbf{F}_{z2}}{\partial \mathbf{y}_1} & 0 & \cdots & \frac{\partial \mathbf{F}_{z2}}{\partial \mathbf{y}_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \mathbf{F}_{zn}}{\partial \mathbf{y}_1} & \frac{\partial \mathbf{F}_{zn}}{\partial \mathbf{y}_2} & \cdots & 0 \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} \frac{\partial \mathbf{F}_{z1}}{\partial \mathbf{x}_s} \\ \frac{\partial \mathbf{F}_{z2}}{\partial \mathbf{x}_s} \\ \cdots \\ \frac{\partial \mathbf{F}_{zn}}{\partial \mathbf{x}_s} \end{bmatrix},$$

$$\mathbf{G} = \begin{bmatrix} \frac{\partial \mathbf{F}_{z1}}{\partial \mathbf{x}_1} & 0 & \cdots & 0 \\ 0 & \frac{\partial \mathbf{F}_{z2}}{\partial \mathbf{x}_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \frac{\partial \mathbf{F}_{zn}}{\partial \mathbf{x}_n} \end{bmatrix}, \quad \Delta\mathbf{z} = \begin{bmatrix} \mathbf{z}_1 - \boldsymbol{\mu}_{z1} \\ \mathbf{z}_2 - \boldsymbol{\mu}_{z2} \\ \cdots \\ \mathbf{z}_n - \boldsymbol{\mu}_{zn} \end{bmatrix},$$

and $\mathbf{H} = \begin{bmatrix} \mathbf{e}_{z1} - \boldsymbol{\mu}_{ez1} \\ \mathbf{e}_{z2} - \boldsymbol{\mu}_{ez2} \\ \cdots \\ \mathbf{e}_{zn} - \boldsymbol{\mu}_{ezn} \end{bmatrix}$.

Since $\Delta\mathbf{x}_s$, $\Delta\mathbf{x}$, $\mathbf{D}$ and $\mathbf{H}$ in Eqn. (A5) are mutually independent, the variance of the system output can be expressed as

$$\mathbf{D}_z = \mathbf{I}\mathbf{D}_{xs} + \mathbf{J}\mathbf{D}_x + \mathbf{K}\mathbf{D}_{ye} + \mathbf{D}_{ze}, \tag{A6}$$

where $\mathbf{D}_z = \begin{bmatrix} \mathbf{s}_{z1}^2 \\ \mathbf{s}_{z2}^2 \\ \cdots \\ \mathbf{s}_{zn}^2 \end{bmatrix}$, $\mathbf{D}_y = \begin{bmatrix} \mathbf{s}_{y1}^2 \\ \mathbf{s}_{y2}^2 \\ \cdots \\ \mathbf{s}_{yn}^2 \end{bmatrix}$, $\mathbf{D}_x = \mathbf{s}_x^2$,

$$\mathbf{D}_{xs} = \begin{bmatrix} \mathbf{s}_{x1}^2 \\ \mathbf{s}_{x2}^2 \\ \cdots \\ \mathbf{s}_{xn}^2 \end{bmatrix}, \quad \mathbf{D}_{ye} = \begin{bmatrix} \mathbf{s}_{ye1}^2 \\ \mathbf{s}_{ye2}^2 \\ \cdots \\ \mathbf{s}_{yen}^2 \end{bmatrix}, \quad \mathbf{D}_{ze} = \begin{bmatrix} \mathbf{s}_{ze1}^2 \\ \mathbf{s}_{ze2}^2 \\ \cdots \\ \mathbf{s}_{zen}^2 \end{bmatrix},$$

$$\mathbf{I} = \{i_{ij}\}, \quad i_{ij} = \{\mathbf{E}(\mathbf{A}^{-1}\mathbf{B}) + \mathbf{F}\}_{ij}^2,$$

$$\mathbf{J} = \{j_{ij}\}, \quad j_{ij} = \{\mathbf{E}(\mathbf{A}^{-1}\mathbf{C}) + \mathbf{G}\}_{ij}^2$$

$$\mathbf{K} = \{k_{ij}\}, \quad k_{ij} = \{\mathbf{E}\mathbf{A}^{-1}\}_{ij}^2,$$

and all the $\mathbf{s}^2$ are the variance vectors.