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METHODS FOR ROBUST MULTIDISCIPLINARY DESIGN

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Abstract

Robust design has been gaining wide attention and its applications have been extended to making reliable decisions when designing complex engineering systems under a multidisciplinary design environment. Though the usefulness of robust multidisciplinary design is widely acknowledged, its implementation is rare. One of the reasons is due to the complexity and computational burden associated with the evaluation of performance variations caused by the randomness (uncertainty) of a system. In this paper, we develop a robust multidisciplinary design procedure that utilizes efficient methods for uncertainty analysis. Different from the existing uncertainty analysis techniques, our proposed techniques bring the features of MDO framework into consideration. The system uncertainty analysis method (SUAM) and the concurrent subsystem uncertainty analysis method (CSSUAM) are developed to estimate the mean and variance of system performance subject to uncertainties associated with both design parameters and design models. The techniques used for uncertainty analysis will significantly reduce the amount of design evaluations at the system level, and therefore improve the efficiency of robust design in the domain of MDO. The merits and limitations of the proposed techniques are illustrated through example problems.

1. Introduction

Robust design has been gaining wide attention and its applications have been extended from improving the quality of individual components to designing complex engineering systems. The methods for robust design have progressed from the initial Taguchi's "parameter design method" [1] to recent nonlinear programming methods [2-4] that formulate robust design problems as nonlinear optimization problems with multiple objectives subject to feasibility robustness. Based on its fundamental principle, i.e., improving the quality of a product by minimizing the effects of variation without eliminating the causes [1], robust design has become one of the powerful tools to assist designers to make reliable decisions under uncertainties [5]. Multidisciplinary Design Optimization (MDO) is another useful tool for designing complex systems [6-8]. MDO focuses on optimizing the performance and reducing the costs of complex systems involving multiple interacting disciplines, such as those found in aircraft, spacecraft, automobiles, and industrial manufacturing applications. It is generally recognized that there always exist uncertainties in any engineering systems due to variations in design conditions and predictions used in mathematical models [9]. However, even though we have seen many applications of MDO, the treatment of uncertainties under multidisciplinary design has received very limited attention [10,11]. There is a great potential to integrate the robust design concept and the MDO framework for rational decision-making in designing complex systems.

In recent developments, some preliminary results of robust design for MDO are reported [11, 12, 14]. In these works, response surface models for system level objective and constraints are created to replace the computationally expensive simulation models. Based on the simplified models, the mean and variance of the system behaviors are evaluated through uncertainty analysis and then utilized to obtain the robust optimal solutions. When conducting uncertainty analysis, most of these approaches utilize design evaluations at the system level, namely, the all-in-one approach.

Our aim in this paper is to integrate the robust design concept with MDO by applying efficient methods for uncertainty analysis that bring the features of MDO framework into consideration. The developed methods will assist designers to make reliable decisions when there are uncertainties associated with both design parameters and design models. The techniques used for uncertainty analysis will significantly reduce

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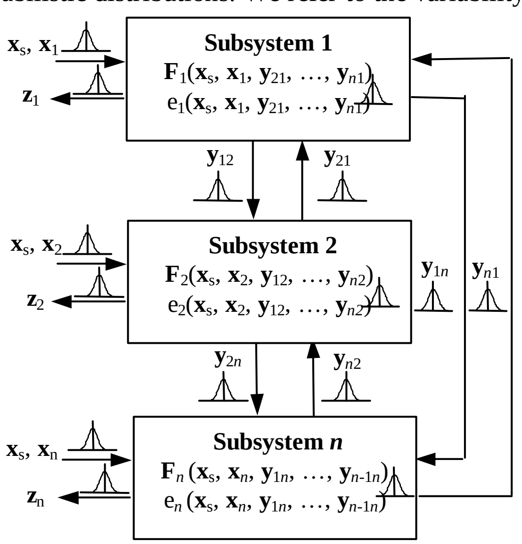
the amount of design evaluations at the system level, and therefore improve the efficiency of robust design in the domain of MDO.

2. ROBUST MULTIDISCIPLINARY DESIGN UNDER UNCERTAINTIES

2.1 Uncertainties in Multidisciplinary Design Optimization

Consider an n-discipline system shown in Fig.1 as the basis for discussion, where each box represents a subsystem model for design evaluation. $\mathbf{x}_s$ are the design variables (parameters) used in more than one subsystems. They are termed as sharing variables in the literature [7]. $\mathbf{x}_i$ $(i = 1,2,\cdots n)$ are the local design variables (parameters) of subsystem *i*. Note that $\mathbf{x}_s$ and $\mathbf{x}_i$ are mutually exclusive. In general, $\mathbf{x}_s$ and $\mathbf{x}_i$ may involve uncertainties that can be described by probabilistic distributions. We refer to the variability of



$\mathbf{F}_i=\{\mathbf{F}_{zi}, \mathbf{F}_{yi}\}$, $\mathbf{e}_i=\{\mathbf{e}_{zi}, \mathbf{e}_{yi}\}$

Figure 1. Coupled system

model input variables as "*external uncertainty*". All the elements of the vectors $\mathbf{x}_s$ and $\mathbf{x}$ are assumed to be independent.

$\mathbf{y}_{ij}(i \neq j)$ are linking variables evaluated as outputs of subsystem *i*, at the same time, required as inputs to subsystem *j*. We denote $\mathbf{y}_i = \{\mathbf{y}_{ij} \mid j = 1,2,\cdots,n,\ j \neq i\}$ for the set of linking variables generated as outputs from subsystem *i* and taken as inputs to the other subsystems and $\mathbf{y}^i = \{\mathbf{y}_1,\cdots \mathbf{y}_{i-1},\mathbf{y}_{i+1},\cdots,\mathbf{y}_n\}$ for the set of linking variables taken as inputs to subsystem *i*.

For subsystem *i*, $\mathbf{y}_i$ is expressed as

$$\mathbf{y}_i = \mathbf{F}_{yi}(\mathbf{x}_s, \mathbf{x}_i, \mathbf{y}^i)\,. \tag{1}$$

We denote the general outputs of subsystem *i* as $\mathbf{z}_i\,(i = 1,2,\cdots,n)$, which are expressed as

$$\mathbf{z}_i = \mathbf{F}_{zi}(\mathbf{x}_s, \mathbf{x}_i, \mathbf{y}^i)\,. \tag{2}$$

The general outputs of each subsystem $\mathbf{z}_i$ are often associated with the system attributes for the evaluations of constraints and objectives in optimization.

To capture the uncertainty associated with the errors inherent in the mathematical models $\mathbf{F}_{yi}(\mathbf{x}_s, \mathbf{x}_i, \mathbf{y}^i)$ and $\mathbf{F}_{zi}(\mathbf{x}_s, \mathbf{x}_i, \mathbf{y}^i)$, the error models $\mathbf{e}_{yi}(\mathbf{x}_s, \mathbf{x}_i, \mathbf{y}^i)$ and $\mathbf{e}_{zi}(\mathbf{x}_s, \mathbf{x}_i, \mathbf{y}^i)$ are introduced. We refer to this type of uncertainty as "*internal uncertainty*". The linking variables and the general outputs of subsystem *i* can then be derived as

$$\mathbf{y}_i = \mathbf{F}_{yi}(\mathbf{x}_s, \mathbf{x}_i, \mathbf{y}^i) + \mathbf{e}_{yi}(\mathbf{x}_s, \mathbf{x}_i, \mathbf{y}^i)\,, \tag{3}$$

and

$$\mathbf{z}_i = \mathbf{F}_{zi}(\mathbf{x}_s, \mathbf{x}_i, \mathbf{y}^i) + \mathbf{e}_{zi}(\mathbf{x}_s, \mathbf{x}_i, \mathbf{y}^i)\,. \tag{4}$$

For simplicity, we assume all the error models $\mathbf{e}_{yi}(\cdot)$ and $\mathbf{e}_{zi}(\cdot)$ are the functions of the mean values of $\mathbf{x}_s, \mathbf{x}_i, \mathbf{y}^i$ only.

2.2 Formulation of Robust Multidisciplinary Optimization

From the viewpoint of robust design, the goal of a design is to make the system (or product) inert to the potential variations without eliminating the sources of uncertainty [1]. The same concept is used here to reduce the impact of both external and internal uncertainties associated with the mathematical models used in MDO. The robust optimization objective is achieved by simultaneously "optimizing the mean performance" and "reducing the performance variation", subject to the robustness of constraints (feasibility robustness) [9]. Let $\mathbf{a}(\mathbf{x}_s, \mathbf{x})$ and $\mathbf{g}(\mathbf{x}_s, \mathbf{x})$ be the objective and constraints of a system, respectively, the general form of the objective can be expressed as

$$\min \quad [\boldsymbol{\mu}_\mathbf{a}(\mathbf{x}_s, \mathbf{x}), \mathbf{s}_\mathbf{a}(\mathbf{x}_s, \mathbf{x})]\,, \tag{5}$$

where $\boldsymbol{\mu}_\mathbf{a}(\mathbf{x}_s, \mathbf{x})$ and $\mathbf{s}_\mathbf{a}(\mathbf{x}_s, \mathbf{x})$ are the mean value and the standard deviation of $\mathbf{a}(\mathbf{x}_s, \mathbf{x})$, respectively. Feasibility robustness can be achieved by increasing the values of constraint functions by the amount of functional variations [16] as

$$\boldsymbol{\mu}_\mathbf{g} + k\mathbf{s}_\mathbf{g} \leq 0\,, \tag{6}$$

where $\boldsymbol{\mu}_\mathbf{g}$ and $\mathbf{s}_\mathbf{g}$ are the mean values and the standard deviations of $\mathbf{g}(\mathbf{x}_s, \mathbf{x})$. *k* is a constant related to the probability of constraint satisfaction. For example, when $\mathbf{g}(\mathbf{x}_s, \mathbf{x})$ follows the normal distribution, $k = 1$ corresponds to the probability ≈ 0.8413, $k = 2$ the probability ≈ 0.9772, and so on. The robust design model is summarized as shown in Fig. 2:

Given:	Distributions of external and internal uncertainties
Find:	Robust design decisions ($\mathbf{x}_s$ and $\mathbf{x}$)
Subject to:	System constraints: $\boldsymbol{\mu}_{\mathbf{g}} + k\mathbf{s}_{\mathbf{g}} \le 0$
Objectives:	a. Optimize the mean of system attributes $\boldsymbol{\mu}_{\mathbf{a}}(\mathbf{x}_s, \mathbf{x})$ b . Minimize the standard deviation of system attributes $\mathbf{s}_{\mathbf{a}}(\mathbf{x}_s, \mathbf{x})$

Figure 2. Robust Design Formulation

As a part of system output $\mathbf{z}$, the mean values $\boldsymbol{\mu}_{\mathbf{a}}$ and $\boldsymbol{\mu}_{\mathbf{g}}$ can be approximated as

$$\boldsymbol{\mu}_{\mathbf{z}} = \mathbf{F}_{\mathbf{z}i}(\boldsymbol{\mu}_s, \boldsymbol{\mu}_{\mathbf{x}_i}, \boldsymbol{\mu}_z^i) + \boldsymbol{\mu}_{\mathbf{e}_{zi}}, \tag{7}$$

with $\boldsymbol{\mu}_{\mathbf{y}_i} = \mathbf{F}_{\mathbf{y}i}(\boldsymbol{\mu}_s, \boldsymbol{\mu}_{\mathbf{x}_i}, \boldsymbol{\mu}_y^i) + \boldsymbol{\mu}_{\mathbf{e}_{yi}}$, (8)

where $\boldsymbol{\mu}_{\mathbf{ez}i}$ and $\boldsymbol{\mu}_{\mathbf{ey}i}$ are the mean values of $\mathbf{e}_{\mathbf{z}i}$ and $\mathbf{e}_{\mathbf{y}i}$, respectively.

The evaluations of standard deviations $\mathbf{s}_{\mathbf{a}}$ and $\mathbf{s}_{\mathbf{g}}$ are more complicated and computationally intensive compared to those of mean values. Efficient methods for uncertainty analysis have been developed to bring the features of MDO framework into consideration. We will provide the details in the next section.

2.3 Uncertainty Analysis under the MDO Framework

Two methods have been developed to evaluate the variance of the end performance of a multidisciplinary design system, given the descriptions of both internal and external uncertainties. One is called the system uncertainty analysis method (SUAM) and the other concurrent subsystem uncertainty analysis method (CSSUAM).

2.3.1 System Uncertainty Analysis Method (SUAM)

The SUAM is an approach that utilizes Taylor approximations as well as sensitivity analysis (first-order derivatives) to evaluate the mean and variance of system attributes subject to both internal and external uncertainties in a multidisciplinary system. To obtain the variances of system outputs, linking variables $\mathbf{y}_i$ $(i = 1,2,\cdots n)$ are first linearized by first-order Taylor approximations at the system level and then derived simultaneously based on a set of linear equations. Next, within each individual subsystem, the system outputs $\mathbf{z}_i$ $(i = 1,2,\cdots n)$ are linearized by the first-order Taylor approximations in terms of inputs $\mathbf{x}_s$, $\mathbf{x}_i$, and the linking variables $\mathbf{y}_i$. Finally, the variances of system outputs are obtained based on the linearized formulations of $\mathbf{z}_i$. These are further explained as the following.

From Eqn. (1), the linking variables $\mathbf{y}_i$ are approximated using Taylor's expansion as

$$\Delta\mathbf{y}_i = \sum_{\substack{j=1 \\ j\ne i}}^{n} \frac{\partial \mathbf{F}_{\mathbf{y}i}}{\partial \mathbf{y}_j}\Delta\mathbf{y}_j + \frac{\partial \mathbf{F}_{\mathbf{y}i}}{\partial \mathbf{x}_s}\Delta\mathbf{x}_s + \frac{\partial \mathbf{F}_{\mathbf{y}i}}{\partial \mathbf{x}_i}\Delta\mathbf{x}_i + \Delta\mathbf{e}_{\mathbf{y}i}, \tag{9}$$

$(i = 1,2,\cdots,n)$

which yields

$$\Delta\mathbf{y} = \mathbf{A}^{-1}\mathbf{B}\Delta\mathbf{x}_s + \mathbf{A}^{-1}\mathbf{C}\Delta\mathbf{x} + \mathbf{A}^{-1}\mathbf{D}, \tag{10}$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{I}_1 & -\frac{\partial \mathbf{F}_{\mathbf{y}1}}{\partial \mathbf{y}_2} & \cdots & -\frac{\partial \mathbf{F}_{\mathbf{y}1}}{\partial \mathbf{y}_n} \\ -\frac{\partial \mathbf{F}_{\mathbf{y}2}}{\partial \mathbf{y}_1} & \mathbf{I}_2 & \cdots & -\frac{\partial \mathbf{F}_{\mathbf{y}2}}{\partial \mathbf{y}_n} \\ \cdots & \cdots & \cdots & \cdots \\ -\frac{\partial \mathbf{F}_{\mathbf{y}n}}{\partial \mathbf{y}_1} & -\frac{\partial \mathbf{F}_{\mathbf{y}n}}{\partial \mathbf{y}_2} & \cdots & \mathbf{I}_n \end{bmatrix}$$

($\mathbf{I}_i$ $(i = 1,2,\cdots,n)$ are the identity matrixes),

$$\mathbf{B} = \begin{bmatrix} \frac{\partial \mathbf{F}_{\mathbf{y}1}}{\partial \mathbf{x}_s} \\ \frac{\partial \mathbf{F}_{\mathbf{y}2}}{\partial \mathbf{x}_s} \\ \cdots \\ \frac{\partial \mathbf{F}_{\mathbf{y}n}}{\partial \mathbf{x}_s} \end{bmatrix}, \ \mathbf{C} = \begin{bmatrix} \frac{\partial \mathbf{F}_{\mathbf{y}1}}{\partial \mathbf{x}_1} & 0 & \cdots & 0 \\ 0 & \frac{\partial \mathbf{F}_{\mathbf{y}2}}{\partial \mathbf{x}_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \frac{\partial \mathbf{F}_{\mathbf{y}n}}{\partial \mathbf{x}_n} \end{bmatrix},$$

$$\mathbf{D} = \begin{bmatrix} \mathbf{e}_{\mathbf{y}1} - \boldsymbol{\mu}_{\mathbf{ey}1} \\ \mathbf{e}_{\mathbf{y}2} - \boldsymbol{\mu}_{\mathbf{ey}2} \\ \cdots \\ \mathbf{e}_{\mathbf{y}n} - \boldsymbol{\mu}_{\mathbf{ey}n} \end{bmatrix}, \ \Delta\mathbf{x}_s = \mathbf{x}_s - \boldsymbol{\mu}_{\mathbf{x}s}, \Delta\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 - \boldsymbol{\mu}_{\mathbf{x}1} \\ \mathbf{x}_2 - \boldsymbol{\mu}_{\mathbf{x}2} \\ \cdots \\ \mathbf{x}_n - \boldsymbol{\mu}_{\mathbf{x}n} \end{bmatrix},$$

and $\Delta\mathbf{y} = \begin{bmatrix} \mathbf{y}_1 - \boldsymbol{\mu}_{\mathbf{y}1} \\ \mathbf{y}_2 - \boldsymbol{\mu}_{\mathbf{y}2} \\ \cdots \\ \mathbf{y}_n - \boldsymbol{\mu}_{\mathbf{y}n} \end{bmatrix}$.

Similarly, we can approximate the general system outputs as

$$\Delta \mathbf{z}_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{\partial \mathbf{F}_{zi}}{\partial \mathbf{y}_j} \Delta \mathbf{y}_j + \frac{\partial \mathbf{F}_{zi}}{\partial \mathbf{x}_s} \Delta \mathbf{x}_s + \frac{\partial \mathbf{F}_{zi}}{\partial \mathbf{x}_i} \Delta \mathbf{x}_i + \Delta \mathbf{e}_{zi} . \quad (11)$$

The reorganization of the above equation yields

$$\begin{aligned} \Delta \mathbf{z} &= \mathbf{E} ?\, \mathbf{y} + \mathbf{F} \Delta \mathbf{x}_s + \mathbf{G} \Delta \mathbf{x} + \mathbf{H} \\ &= [\mathbf{E}(\mathbf{A}^{-1}\mathbf{B}) + \mathbf{F}] \Delta \mathbf{x}_s + [\mathbf{E}(\mathbf{A}^{-1}\mathbf{C}) + \mathbf{G}] \Delta \mathbf{x} \\ &\quad + \mathbf{E}\mathbf{A}^{-1}\mathbf{D} + \mathbf{H} , \end{aligned} \quad (12)$$

where

$$\mathbf{E} = \begin{bmatrix} 0 & \frac{\partial \mathbf{F}_{z1}}{\partial \mathbf{y}_2} & \cdots & \frac{\partial \mathbf{F}_{z1}}{\partial \mathbf{y}_n} \\ \frac{\partial \mathbf{F}_{z2}}{\partial \mathbf{y}_1} & 0 & \cdots & \frac{\partial \mathbf{F}_{z2}}{\partial \mathbf{y}_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \mathbf{F}_{zn}}{\partial \mathbf{y}_1} & \frac{\partial \mathbf{F}_{zn}}{\partial \mathbf{y}_2} & \cdots & 0 \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} \frac{\partial \mathbf{F}_{z1}}{\partial \mathbf{x}_s} \\ \frac{\partial \mathbf{F}_{z2}}{\partial \mathbf{x}_s} \\ \cdots \\ \frac{\partial \mathbf{F}_{zn}}{\partial \mathbf{x}_s} \end{bmatrix},$$

$$\mathbf{G} = \begin{bmatrix} \frac{\partial \mathbf{F}_{z1}}{\partial \mathbf{x}_1} & 0 & \cdots & 0 \\ 0 & \frac{\partial \mathbf{F}_{z2}}{\partial \mathbf{x}_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \frac{\partial \mathbf{F}_{zn}}{\partial \mathbf{x}_n} \end{bmatrix}, \quad \Delta \mathbf{z} = \begin{bmatrix} \mathbf{z}_1 - \bar{\mathbf{z}}_1 \\ \mathbf{z}_2 - \bar{\mathbf{z}}_2 \\ \cdots \\ \mathbf{z}_n - \bar{\mathbf{z}}_n \end{bmatrix},$$

and $\mathbf{H} = \begin{bmatrix} \mathbf{e}_{z1} - \bar{\mathbf{e}}_{z1} \\ \mathbf{e}_{z2} - \bar{\mathbf{e}}_{z2} \\ \cdots \\ \mathbf{e}_{zn} - \bar{\mathbf{e}}_{zn} \end{bmatrix}$.

Since $\Delta \mathbf{x}_s$, $\Delta \mathbf{x}$, $\mathbf{D}$ and $\mathbf{H}$ in Eqn. (12) are mutually independent, the variance of a general system output can be expressed as

$$\mathbf{D}_z = \mathbf{I}\mathbf{D}_{xs} + \mathbf{J}\mathbf{D}_x + \mathbf{K}\mathbf{D}_{ye} + \mathbf{D}_{ze} , \quad (13)$$

where

$$\mathbf{D}_z = \begin{bmatrix} \mathbf{s}_{z1}^2 \\ \mathbf{s}_{z2}^2 \\ \cdots \\ \mathbf{s}_{zn}^2 \end{bmatrix}, \quad \mathbf{D}_y = \begin{bmatrix} \mathbf{s}_{y1}^2 \\ \mathbf{s}_{y2}^2 \\ \cdots \\ \mathbf{s}_{yn}^2 \end{bmatrix}, \quad \mathbf{D}_x = \mathbf{s}_x^2, \quad \mathbf{D}_{xs} = \begin{bmatrix} \mathbf{s}_{x1}^2 \\ \mathbf{s}_{x2}^2 \\ \cdots \\ \mathbf{s}_{xn}^2 \end{bmatrix},$$

$$\mathbf{D}_{ye} = \begin{bmatrix} \mathbf{s}_{ye1}^2 \\ \mathbf{s}_{ye2}^2 \\ \cdots \\ \mathbf{s}_{yen}^2 \end{bmatrix}, \quad \mathbf{D}_{ze} = \begin{bmatrix} \mathbf{s}_{ze1}^2 \\ \mathbf{s}_{ze2}^2 \\ \cdots \\ \mathbf{s}_{zen}^2 \end{bmatrix},$$

$$\mathbf{I} = \{i_{ij}\}, \; i_{ij} = \{\mathbf{E}(\mathbf{A}^{-1}\mathbf{B}) + \mathbf{F}\}_{ij}^2 ,$$

$$\mathbf{J} = \{j_{ij}\}, \; j_{ij} = \{\mathbf{E}(\mathbf{A}^{-1}\mathbf{C}) + \mathbf{G}\}_{ij}^2$$

$$\mathbf{K} = \{k_{ij}\}, \; k_{ij} = \{\mathbf{E}\mathbf{A}^{-1}\}_{ij}^2 ,$$

and all the $\mathbf{s}^2$ are the variance vectors.

From Eqn. (13), it is noted that with this approach the total variation of a system output can be derived as the sum of the variations contributed by four individual sources, i.e., the sharing variables $\mathbf{x}_s$, the subsystem variables $\mathbf{x}$, the linking variable $\mathbf{y}_i$, and the model uncertainty associated with system output $\mathbf{z}_i$. The derivation is based on the independence assumption among all the variances.

2.3.2 Concurrent Subsystem Uncertainty Analysis Method (CSSUAM)

To reduce the computational demand of system level analysis and to make use of the parallel computing technique, a concurrent subsystem uncertainty analysis method (CSSUAM) is developed to estimate the mean and variance of end system performance. Uncertainty analysis is carried out locally within each subsystem, and the compatibility among subsystems is achieved by matching the linking variables $\mathbf{y}_i$. As shown in Fig. 3, the compatibility is realized by a system level optimizer which sets the target values of the mean and variance of the linking variables and minimizes the deviations between the targets and those that are actually generated through the subsystems analyses. The deviations are minimized to zero when the compatibility is achieved.

If neglecting the dependence between the variances of $\mathbf{y}_i$ and $\mathbf{x}_s, \mathbf{x}_i$, $\mathbf{y}_j (j \neq i)$, the variance $\bar{\mathbf{s}}_{yi}^{*}$ of $\mathbf{y}_i$ can be derived from Eqn. (3) as

$$\mathbf{s}_{yi}^{*2} = \sum_{\substack{j=1 \\ j \neq i}}^{n} \left(\frac{\partial \mathbf{F}_{yi}}{\partial \mathbf{y}_j} \right)^2 \mathbf{s}_{yj}^2 + \left(\frac{\partial \mathbf{F}_{yi}}{\partial \mathbf{x}_s} \right)^2 \mathbf{s}_{xs}^2 + \left(\frac{\partial \mathbf{F}_{yi}}{\partial \mathbf{x}_i} \right)^2 \mathbf{s}_{xi}^2 + \mathbf{s}_{eyi}^2 \quad (14)$$

Once the mean values and variances of $\mathbf{y}_i$ are obtained using the optimization shown in Fig. 3, if neglecting the dependence between the variances of $\mathbf{z}_i$ and $\mathbf{x}_s, \mathbf{x}_i$, $\mathbf{y}_i$, the variance of $\mathbf{z}_i$ can be derived from Eqn. (11) as:

$$\mathbf{s}_{zi}^2 = \sum_{j=1}^{n} \left(\frac{\partial \mathbf{F}_{zi}}{\partial \mathbf{y}_j} \right)^2 \mathbf{s}_{yj}^2 + \left(\frac{\partial \mathbf{F}_{zi}}{\partial \mathbf{x}_s} \right)^2 \mathbf{s}_{xs}^2 + \left(\frac{\partial \mathbf{F}_{zi}}{\partial \mathbf{x}_i} \right)^2 \mathbf{s}_{xi}^2 + \mathbf{s}_{ezi}^2 \quad (15)$$

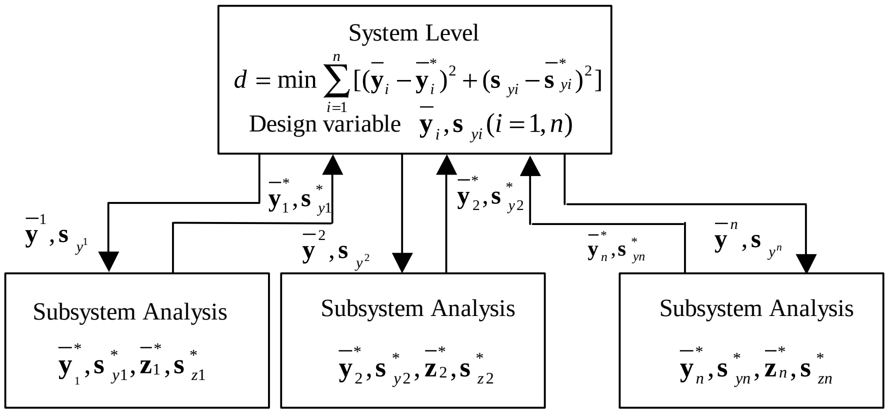


Figure 3. Concurrent Subsystem Uncertainty Analysis

2.4 Implementation of Robust Multidisciplinary Design

Based on the robust design formulation presented in Section 2.2 and the proposed uncertainty analysis methods in Section 2.3, a robust multidisciplinary design procedure is developed as shown in Fig. 4. Given the probabilistic descriptions of uncertain information, the procedure will first check whether the models for subsystems $\mathbf{F}_{yi}(\mathbf{x}_s, \mathbf{x}_i, \mathbf{y}^i)$ and $\mathbf{F}_{zi}(\mathbf{x}_s, \mathbf{x}_i, \mathbf{y}^i)$ are computationally expensive to evaluate. If yes, surrogate models (for example, response surface models) can be created to replace the initial expensive models. Obviously, using surrogate models will introduce additional uncertainty (*internal uncertainty*). If the errors of surrogate models can be estimated, the error model of internal uncertainty will be modified accordingly. When implementing optimization for robust design, the methods for uncertainty analysis as those introduced in Sections 2.2 and 2.3 are used.

3. Example Problems

Two mathematical problems are used in this work to illustrate the tangible effects of the proposed approach. The problems are constructed to have the challenging features of a MDO problem, such as highly nonlinear behaviors, strong couplings, multiple systems, stochastic design variables, and uncertainty of models. We note that the partitioning into "disciplines" is artificial in both examples. The second problem is

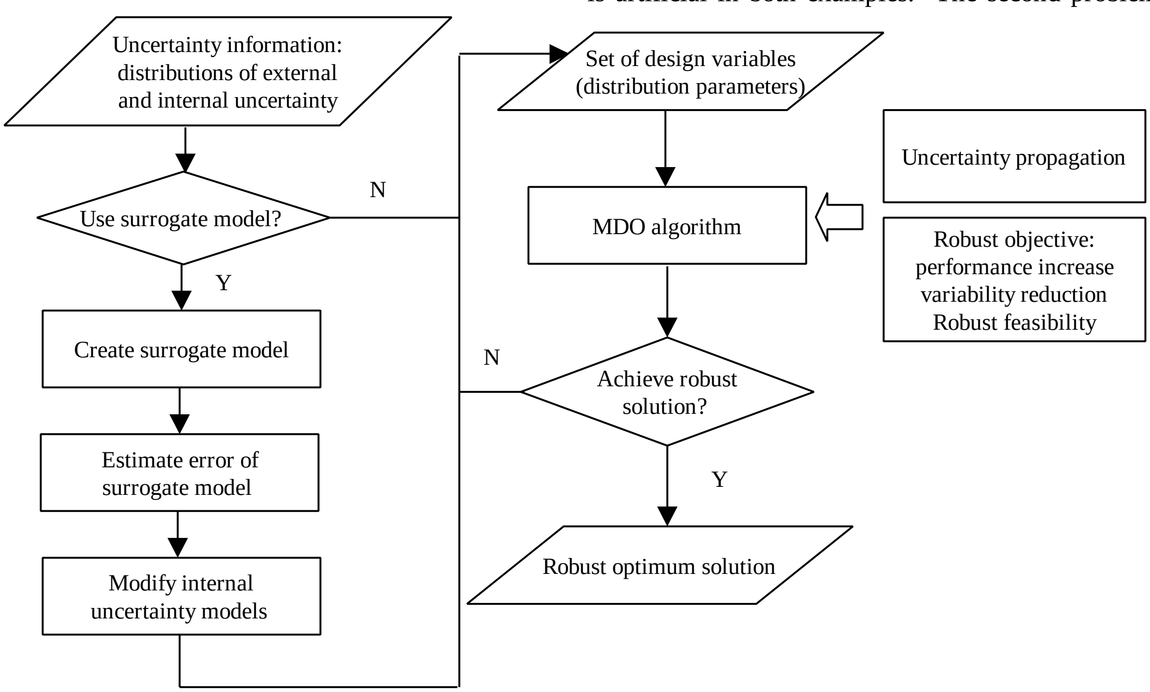


Figure 4. Procedure of Robust Multidisciplinary Optimization

slightly different from the first one in that the end system performance function z is more dependent on the linking variables. In both cases, the constraints are constructed to bound the design variables and make the problems highly constrained.

3.1 Example 1

Two subsystems are considered in this example. Fig. 5 is an information flow diagram of the example. The functional relationships are represented as:

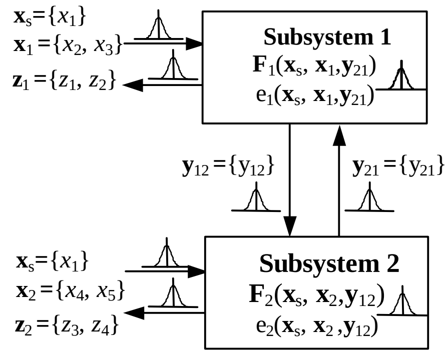
$$\mathbf{x}_s = \{x_1\},\ \mathbf{x}_1 = \{x_2, x_3\},\ \mathbf{x}_2 = \{x_4, x_5\}$$
$$\mathbf{y}_1 = \mathbf{y}_{12} = \{y_{12}\},\ \mathbf{y}_2 = \mathbf{y}_{21} = \{y_{21}\}$$
$$\mathbf{z}_1 = \{z_1, z_2\},\ \mathbf{z}_2 = \{z_3, z_4\}$$
$$\mathbf{e}_{y1}(\mathbf{x}_s, \mathbf{x}_1, \mathbf{y}_2) = \{e_{y12}(\mathbf{x}_s, \mathbf{x}_1, \mathbf{y}_2)\}$$
$$\mathbf{e}_{y2}(\mathbf{x}_s, \mathbf{x}_2, \mathbf{y}_1) = \{e_{y21}(\mathbf{x}_s, \mathbf{x}_2, \mathbf{y}_1)\}$$
$$\mathbf{e}_{z1}(\mathbf{x}_s, \mathbf{x}_1, \mathbf{y}_2) = \{e_{z1}(\mathbf{x}_s, \mathbf{x}_1, \mathbf{y}_2), e_{z2}(\mathbf{x}_s, \mathbf{x}_1, \mathbf{y}_2)\}$$
$$\mathbf{e}_{z2}(\mathbf{x}_s, \mathbf{x}_2, \mathbf{y}_1) = \{e_{z3}(\mathbf{x}_s, \mathbf{x}_2, \mathbf{y}_1), e_{z4}(\mathbf{x}_s, \mathbf{x}_2, \mathbf{y}_1)\}$$
$$e_{z2}(\mathbf{x}_s, \mathbf{x}_1, \mathbf{y}_2) = e_{y21}(\mathbf{x}_s, \mathbf{x}_1, \mathbf{y}_2)$$
$$e_{z4}(\mathbf{x}_s, \mathbf{x}_2, \mathbf{y}_1) = e_{y12}(\mathbf{x}_s, \mathbf{x}_2, \mathbf{y}_1)$$
$$\mathbf{F}_{y12}(\mathbf{x}_s, \mathbf{x}_1, \mathbf{y}_1) = \{F_{y12}(\mathbf{x}_s, \mathbf{x}_1, \mathbf{y}_1)\}$$
$$= x_1^2 + 2x_2 - x_3 + 2\sqrt{y_{21}} - 10$$
$$\mathbf{F}_{z1}(\mathbf{x}_s, \mathbf{x}_1, \mathbf{y}_2) = \{F_{z1}(\mathbf{x}_s, \mathbf{x}_1, \mathbf{y}_2), F_{z2}(\mathbf{x}_s, \mathbf{x}_1, \mathbf{y}_2)\}$$
$$F_{z1}(\mathbf{x}_s, \mathbf{x}_1, \mathbf{y}_2) = x_1^2 + 2x_2 + x_3 + x_2 e^{-y_{21}}$$
$$\mathbf{F}_{y21}(\mathbf{x}_s, \mathbf{x}_2, \mathbf{y}_2) = \{F_{y21}(\mathbf{x}_s, \mathbf{x}_2, \mathbf{y}_2)\}$$
$$= (x_1 x_4 + x_4^2 + x_5 + y_{12})$$
$$\mathbf{F}_{z2}(\mathbf{x}_s, \mathbf{x}_2, \mathbf{y}_2) = \{F_{z3}(\mathbf{x}_s, \mathbf{x}_2, \mathbf{y}_2), F_{z4}(\mathbf{x}_s, \mathbf{x}_2, \mathbf{y}_2)\}$$
$$F_{z3}(\mathbf{x}_s, \mathbf{x}_2, \mathbf{y}_1) = 21 - (\sqrt{x_1} + x_4 + x_5\sqrt{0.4 y_{12}})$$
$$F_{z4}(\mathbf{x}_s, \mathbf{x}_2, \mathbf{y}_1) = y_{12} - 10$$

In this problem, z_1 is used to evaluate the robust design objective and z_2~z_4 are used to evaluate the system constraints. The robust design model is stated as

Find: the mean values $\bar{x}_1$ ~ $\bar{x}_5$
Minimize: $F = w_1 m_{z1} + w_2 s_{z1}$
Subject to: $g_1 = m_{z2} + k s_{z2} \le 0$
$g_2 = m_{z3} + k s_{z3} \le 0$
$g_3 = m_{z4} + k s_{z4} \le 0$

In the above model, equal weights ($w_1 = w_2 = 0.5$) are assigned to optimize the mean performance and minimizing the performance deviation, and k=2 (the probability ≈ 0.9772) is used in the constraints to evaluate feasibility robustness. The parameter and model uncertainties are defined in Table 1.

Both the SUAM and the CSSUAM are used to evaluate the mean and variance of each system output. To illustrate the accuracy of these techniques, we compare the results of *z1* and *z2* obtained at three different design points using SUAM and CSSUAM with those obtained from Monte Carlo simulations (MCS). The size of the MCS is chosen as 10,000, large enough to be considered as the "real solution" for the purpose of confirmation. Random simulations are picked to follow normal distributions of the random variables. From the comparison in Table 2, it is noted that the variance obtained using the SUAM are very close to that from the Monte Carlo simulation. The errors of s_{z1} and s_{z2} are all less than 1%. The CSSUAM also provides a good approximation of the system variance, however it is not as accurate as SUAM.



$\mathbf{F}_1=\{\mathbf{F}_{z1}, \mathbf{F}_{y12}\}$, $\mathbf{F}_2=\{\mathbf{F}_{z2}, \mathbf{F}_{y21}\}$

Figure 5. Information Flow – Example 1

Table 1. Descriptions of Parameter and Model Uncertainties

	Mean value	Standard deviation
x_1	$\bar{x}_1$	$0.1\bar{x}_1$
x_2	$\bar{x}_2$	$0.1\bar{x}_2$
x_3	$\bar{x}_3$	$0.1\bar{x}_3$
x_4	$\bar{x}_4$	$0.1\bar{x}_4$
x_5	$\bar{x}_5$	$0.1\bar{x}_5$
e_{y12}	0	$0.1\bar{y}_{12}$
e_{y21}	0	$0.1\bar{y}_{21}$
e_{z1}	0	$0.1\bar{z}_1$
e_{z4}	0	$0.1\bar{z}_2$

The optimization results from using different methods are compared in Table 3. In addition to the optimal solutions, the mean and variance of each output *z*, the values of objective function (F), and constraint functions (g) are also provided. It is observed that the optimum solution generated by SUAM and CSSUAM are all very close to that from the MCS. Since the design solution from SUAM is closer to that of the MCS, the mean and variance of performance obtained by these two methods are closer. The mean and variance obtained from the SUAM and the CSSUAM are further confirmed by running MCS at the solution x identified by the two tested methods.

From the comparison in Table 4, it is noted that the variances confirmed for SUAM are very close to the reference solutions (from MCS as shown in the last column of Table 3). The CSSUAM also provides a good approximation of the system variances at the optimum point, however they are not as accurate as those from SUAM. For this case study, approximations using both methods are acceptable from the practical viewpoint.

Table 2. Comparison of the Variances of System Outputs (Example 1)

$(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4, \bar{x}_5)$	Method	s^2_{z1}	Error of s^2_{z1}	s^2_{z2}	Error of s^2_{z2}
(1,1,1,1,1)	SUAM	0.250	0.04%	0.2296	-0.86%
	CSSUAM	0.250	0.04%	0.2109	-8.94%
	MCS	0.2499	—	0.2316	—
(2,2,2,2,2)	SUAM	1.840	-0.49%	1.2292	-0.11%
	CSSUAM	1.840	-0.49%	1.1292	-8.23%
	MCS	1.8490	—	1.2305	—
(2,5,2,5,2)	SUAM	4.240	-0.4%	2.7553	0.11%
	CSSUAM	4.240	-0.4%	2.5936	-5.77%
	MCS	4.2572	—	2.7524	—

Table 3. Comparisons of Robust Design Results (Example 1)

Method	SUAM	CSSUAM	MCS
$(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4, \bar{x}_5)$	1, 1, 1.7415, 4.2244, 4.7892, 1	1.000972, 1.000981, 1.771770, 4.332033, 4.656617	1.0, 1.0, 1.735409,4.217265, 4.754966
m_{z1}, s_{z1}	4.7415, 0.5789	4.775678, 0.5828	4.738518, 0.5796
m_{z2}, s_{z2} (y_{21})	40.9098, 7.0454	42.277800, 6.503901	41.017680, 7.001794
m_{z3}, s_{z3}	16.5782, 2.2109	16.557050, 2.222777	16.446140, 2.277659
m_{z4}, s_{z4} (y_{12})	14.0507, 2.0253	14.525640, 2.240409	14.036640, 2.020805
F	2.6602	2.679257	2.659060
g_1, g_2, g_3	-6.5399×10^{-4}, 6.7827×10^{-5}, 3.7121×10^{-5}	-0.2856, 0.04482, -2.604008×10^{-3}	-0.02127, -4.968695×10^{-3}, -1.460733×10^{-3}

Table 4. Confirmation of the Optimum Results using MCS (Example 1)

Method	SUAM	CSSUAM
$(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4, \bar{x}_5)$	1, 1, 1.7415, 4.2244, 4.7892	1.000972, 1.000981, 1.771770, 4.332033, 4.656617
m_{z1}, s_{z1}	4.7498, 0.5774	4.7847, 0.5855
m_{z2}, s_{z2} (y_{21})	41.0579, 7.0789	42.0695, 7.2851
m_{z3}, s_{z3}	16.5467, 2.3012	16.3889, 2.2614
m_{z4}, s_{z4} (y_{12})	14.0376, 2.0326	14.1536, 2.0745
F	2.6636	2.6851
g_1, g_2, g_3	-0.2157, -0.0277, -0.1492	-1.6396, 0.0047, 0.0883

3.2 Example 2

Example 2 is constructed to involve higher dependence between the system output z and the linking variable. The mathematical problem is described as the following with the information flow diagram represented in Fig. 6:

$$\mathbf{x}_s = \{x_1\},\ \mathbf{x}_1 = \{x_2, x_3\},\ \mathbf{x}_2 = \{x_4, x_5\}$$

$$\mathbf{y}_1 = \mathbf{y}_{12} = \{y_{12}\},\ \mathbf{y}_2 = \mathbf{y}_{21} = \{y_{21}\}$$

$$\mathbf{z}_1 = \{z_1\},\ \mathbf{z}_2 = \{z_2\}$$

$$\mathbf{e}_{y1}(\mathbf{x}_s, \mathbf{x}_1, \mathbf{y}_2) = \{e_{y12}(\mathbf{x}_s, \mathbf{x}_1, \mathbf{y}_2)\}$$

$$\mathbf{e}_{y2}(\mathbf{x}_s, \mathbf{x}_2, \mathbf{y}_1) = \{e_{y21}(\mathbf{x}_s, \mathbf{x}_2, \mathbf{y}_1)\}$$

$$\mathbf{e}_{z1}(\mathbf{x}_s, \mathbf{x}_1, \mathbf{y}_2) = \{e_{z1}(\mathbf{x}_s, \mathbf{x}_1, \mathbf{y}_2)\}$$

$$\mathbf{e}_{z2}(\mathbf{x}_s, \mathbf{x}_2, \mathbf{y}_1) = \{e_{z2}(\mathbf{x}_s, \mathbf{x}_2, \mathbf{y}_1)\}$$

$$\mathbf{F}_{y12}(\mathbf{x}_s, \mathbf{x}_1, \mathbf{y}_2) = \{F_{y12}(\mathbf{x}_s, \mathbf{x}_1, \mathbf{y}_2)\}$$

$$= x_1^2 + 2x_2 - x_3 + 2\sqrt{y_{21}}$$

$$\mathbf{F}_{z1}(\mathbf{x}_s,\mathbf{x}_1,\mathbf{y}_2)=\{F_{z1}(\mathbf{x}_s,\mathbf{x}_1,\mathbf{y}_2)\}$$
$$=x_1^2+2x_2+x_3+x_2e^{-y_{21}}$$
$$\mathbf{F}_{y21}(\mathbf{x}_s,\mathbf{x}_2,\mathbf{y}_1)=\{F_{y21}(\mathbf{x}_s,\mathbf{x}_2,\mathbf{y}_1)\}$$
$$=x_1x_4+x_4^2+x_5+y_{12}$$
$$\mathbf{F}_{z2}(\mathbf{x}_s,\mathbf{x}_2,\mathbf{y}_1)=\{F_{z2}(\mathbf{x}_s,\mathbf{x}_2,\mathbf{y}_1)\}$$
$$=\sqrt{x_1}+x_4+x_5y_{12}$$
$$\mathbf{F}_z(z_1,z_2)=z_1-12z_2$$

The probabilistic distributions of input and model uncertainties are the same as those considered for example 1 (see Table 1). The robust design model is formulated as:

Find: the mean values $\bar{x}_1 \sim \bar{x}_5$
Minimize: $F = w_1 m_z + w_2 s_z$
Subject to: $g_1 = m_{y12} - k s_{y12} \ge 8$
$g_2 = m_{y21} + k s_{y21} \le 12$
$x_1, x_2, x_3, x_4, x_5 \ge 0$

$w_1 = w_2 = 0.5$ and $k = 1$ (probability ≈ 0.8413) are assigned for this example.

Similar to the studies implemented for example 1, we first compare the accuracy of the proposed uncertainty analysis method at three different design points, using the results from 10,000 MCS as the reference for confirmation (see Table 5). It is observed that both SUAM and CSSUAM introduce little deviation in mean evaluation when compared with the results from MCS. For the estimation of the standard deviations, the SUAM provides very close values to those from MCS. The relative errors of s_z do not exceed 2% at the studying points for SUAM. However, the errors are much larger for CSSUAM, with the error of the first point close to 19%.

Different uncertainty analysis methods have also resulted in different robust design solutions. A comparison is provided in Table 6. The constraints are all active for this example. The optimal design point identified by SUAM is very close to that from the MCS. Considering the stochastic nature (e.g., different seed numbers for generating random numbers) when using MCS for optimization, this difference can be considered as negligible. MCS is also used to verify the true system performance at the optimal point obtained by SUAM. These verification results can be found in the column of SUAM*. Though the optimum solution identified by CSSUAM is feasible and better for minimizing the objective function, the MCS confirmation reveals that the solution point is indeed not feasible. Specifically, the verified probability that y_{21} is less than 12 turns to be 74.5% while the original constraint is set at 84.1% (k=1) for constraint satisfaction.

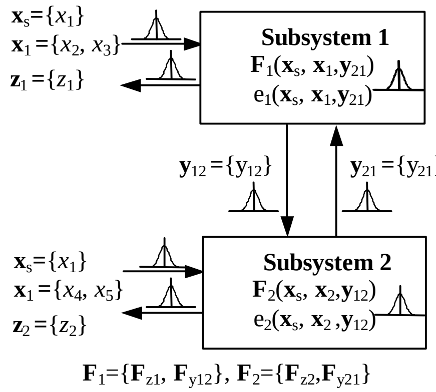


Figure 6. Information Flow (Example 2)

3.3 Discussions

Compared to the use of MCS for uncertainty analysis, both the SUAM and CSSUAM can significantly improve the computational efficiency of robust design in MDO applications. For the evaluation of the mean values of system outputs, only one analysis at the system level is required when using the SUAM. No analysis at the system level is required if using the CSSUAM. For the calculation of variance, if neglecting the evaluation of the first-order derivatives, SUAM only needs one analysis at the system level for solving the simultaneous linear equations to obtain the linear approximations of linking variables $\Delta\mathbf{y}$. The amount of calculation thus depends on the number of linking variables. A small number of linking variables and design variables is preferred to deal with the inversion of matrix **A** and to conduct analysis at the system level. This amount of evaluation required for SUAM is also associated with the total number of design variables $\mathbf{x}_s$ and $\mathbf{x}_i$ (i=1,…, n) for derivative evaluations. When the number of linking variables and the total number of input variables is large, to reduce the computational effort, the CSSUAM can be employed. The CSSUAM does not involve any system level analysis. All the calculations are implemented at the subsystems level, where the dimensions can be significantly reduced compared to the SUAM. On the other hand, the number of subsystem analysis may increase due to the optimization process at the top level (see Fig. 3).

Table 5. Comparison of the Results of Uncertainty Analysis (Example 2)

$(\bar{x}_1,\bar{x}_2,\bar{x}_3,\bar{x}_4,\bar{x}_5)$	Method	m_z	Relative Error	s_z	Relative Error
(.95,.23,.6,.48,.89)	SUAM	-84.03	-0.36%	16.74	1.24%
	CSSUAM	-84.03	-0.36%	13.82	18.44%
	MCS	-83.73	-	16.95	-
(7.3,1.7,4.0,0.6,9.3)	SUAM	-7919	0.74%	1974	0.79%
	CSSUAM	-7919	0.74%	1831	8.01%
	MCS	-7978	-	1990	-
(10,10,10,10,10)	SUAM	-17768	0.69%	4104	0.75%
	CSSUAM	-17769	0.69%	3930	4.94%
	MCS	-17891	-	4135	-

Table 6. Comparison of the Robust Optimal Solutions (Example 2)

Method	MCS	SUAM	SUAM*	CSSUAM	CSSUAM*
Objective	-21.62	-22.04	-21.91	-35.03	-33.61
$\bar{x}_1$	0.9563	0.9359		0.9337	
$\bar{x}_2$	1.192	1.176		1.252	
$\bar{x}_3$	0.000	0.000		0.000	
$\bar{x}_4$	0.000	0.000		0.002	
$\bar{x}_5$	0.3292	0.4052		0.645	
m_z	-46.22	-54.73	-54.53	-84.69	-84.25
s_z	2.973	10.65	10.71	14.65	17.03
y_{12}	9.570	9.532	9.50	9.877	9.826
y_{21}	9.903	9.937	9.91	10.52	10.476

Through two examples, one with less dependence between the system performance and the linking variable than the other, we find the application of SUAM in robust design significantly increases the computational efficiency with little loss of precision in all cases. Though efficient, CSSUAM is less accurate for problems with strong dependence between the system performance and the linking variable. The reason is that the CSSUAM is derived based on the assumption of independence between the variances of $\mathbf{y}_i$ and $\mathbf{x}_s$, $\mathbf{x}_i$, $\mathbf{y}_j\,(j \neq i)$ and the independence between the variances of $\mathbf{z}_i$ and $\mathbf{x}_s$, $\mathbf{x}_i$, $\mathbf{y}_i$ (see Eqns. 15 and 16), while correlations actually exist. We note that the errors of the SUAM and the CSSUAM also come from the Taylor expansion (linearization of system output).

4. CLOSURE

A methodology for integrating the robust design concept into MDO framework is proposed in this paper. To implement the proposed approach, efficient methods for uncertainty analysis are developed in the context of multidisciplinary design that involves external and internal uncertainties. Both the SUAM and the CSSUAM are developed considering the features of the MDO framework. Our approach can be easily integrated with a system optimization process, where the sensitivity information can be shared between uncertainty analysis and optimization search. Though demonstrated for mathematical problems, the concepts and principles presented in this paper can be extended to more complicated systems.

The comparison between the SUAM and the CSSUAM is provided in Section 3.3 and will not be repeated here. While our results indicate that SUAM is more robust with respect to its accuracy in robust design, our study also points out the direction for further improving the CSSUAM, i.e., to refine the approximation by considering the correlations between the variances. Under the situation that the system level analysis may require a great deal of computations and the error limit is reasonably large, the presented CSSUAM could still be a viable approach.

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