Piecewise Quadratic Approximation of the Nondominated Set for Bi-Criteria Programs

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Abstract:

A procedure to approximate the nondominated set for general (continuous) bi-criteria programs is proposed. The piecewise approximation is composed of quadratic curves, each of which is developed locally in a neighborhood of a nondominated point of interest and based on primal-dual relationships associated with the weighted-Tchebycheff scalarization of the original problem. The approximating quadratic functions, in which decision maker's preferences are represented, give a closed-form description of the nondominated set. A numerical example is included.

Keywords: bi-criteria programs, nondominated set, approximation

1. Introduction

During the last decade, one of the active research directions in the field of multiple criteria decision making has been focused on approximations of the efficient or nondominated set. In general, the approximation concepts and methods proposed can be grouped into computational and analytical approaches. Among the former, a large family of methods is based on genetic algorithms, see Goldberg (1989), Osyczka and Kundu (1995, 1996), Cheng and Li (1997), and Grignon and Fadel (1997). The use of genetic algorithms enabled researchers to computationally find a set of points representing the nondominated set and their work concentrated on obtaining a good spread of those points in order to provide a better feedback to the decision maker. Other approaches in this class were based on the weighted Tchebycheff scalarization. Helbig (1991) developed an approximation method of the nondominated set for general multiple criteria problems by perturbation of the ordering cone and applied it to a bi-criteria structural optimization problem. Later he proposed a similar procedure specifically for bi-criteria quasi-convex problems (Helbig, 1994). Jahn and Merkel (1992) also developed a weighted Tchebycheff based approximation for general bi-criteria problems and applied it to a structural optimization problem. Payne (1993) proposed an approximation method for bi-criteria problems that was based upon the concept of representing the nondominated set by a finite set of points and a collection of rectangles defined by these points and enclosing the nondominated set. Yang and Goh (1997) proposed a sandwich method for convex bi-criteria problems and established its quadratic convergence. The approximation was achieved using a convex piecewise linear function. A very different method based on the Parameter Space Investigation (Sobol, 1992) was developed by Sobol and Levitan (1997) for general multiple criteria problems. The method employs a crude global search procedure initiated with a set of trial points iteratively approaching the nondominated points. Schandl et al. (2000) developed and implemented a norm-based approach to bi-criteria programs in which the nondominated set is approximated by the unit ball of a polyhedral norm that also serves as a tool for finding nondominated solutions.

Analytical approaches to describe the nondominated set are more challenging as they aim at obtaining an explicit expression (function or formula) representing a good approximation of this set. In this class of methods, the balance set approach of Galperin (1992), demonstrated later by Galperin and Wiecek (1999) was developed for general (possibly nonconvex) multiple criteria problems. Li et al. (1998) proposed approximation for bi-criteria convex problems by means of a hyper-ellipse whose equation provides a good description of the nondominated set and at the same time requires a minimal amount of information in the form of three nondominated points. Tappeta and Renaud (1998) proposed first and second order approximation of the nondominated set based on sensitivity analysis results. Clearly, more research is needed to obtain explicit approximations of the nondominated set and this paper represents an effort in this direction since having such a description seems to be the ultimate goal in the determination of this set.

We focus on approximating the nondominated set of general bi-criteria problems and provide explicit functions representing the relationship between the criteria. We build our approach upon the weighted Tchebycheff scalarization, a well established approach generating nondominated solutions of general multiple criteria programs (see Steuer, 1984), and its dual counterpart recently developed by Tind and Wiecek (1998). Preliminary results on applying the dual approach to a bi-criteria robust design decision making problem were presented in Chen et al. (1999) and Zhang et al. (1999). In this paper, we show how the primal-dual relationship can be used to locally approximate the nondominated set. The advantage of using this relationship is twofold. First, it yields a local approximating quadratic function whose coefficients are based on the input parameters selected by the decision maker for the weighted Tchebycheff approach. Consequently, decision maker's preferences are expressed in the resulting approximation. Second, computational effort needed to obtain a satisfactory approximating function is significantly reduced since only a single scalar coefficient of that function has to be found. Concatenation of the local curves results in the approximation of a portion or the entire nondominated set.

The concept of using quadratic functions as local approximations is well known in nonlinear programming. For example, Newton's method and sequential quadratic programming numerically exploit quadratic approximations in order to find optimal solutions (Bazaraa et al., 1993). In this paper however, we apply the same concept in a very different capacity as we first find a solution and then use it to derive a local approximation.

In Section 2, we formulate the bi-criteria program, briefly review the weighted Tchebychef scalarization as well as its dual problem, and then discuss the relationships between them. We develop the proposed approximation in Section 3 and present an approximation algorithm in Section 4. Section 5 includes a numerical example and conclusions are given in Section 6.

2. Problem formulation

Consider the following bi-criteria program (BCP):

$$\begin{aligned} &\text{minimize} \quad && f(x) = [\, f_1(x), f_2(x)]^T \\ &\text{subject to} \quad && x \in X, \end{aligned} \tag{1}$$

where $f: R^n \rightarrow R^2$ and $f_i(x) \in C^1$, i = 1, 2. The set X of feasible solutions is given by X = $\{x \in R^n, \mid g_j(x) \leq 0, j = 1, 2, ..., J\}$, where $g_j \in C^1$, j = 1, 2, ..., J.

A point $x^* \in X$ is said to be *a (globally) efficient solution* of the BCP if there does not exist another $x \in X$ for which $f_i(x) \leq f_i(x^*)$, for both i = 1, 2, with strict inequality holding for at least one index i. A point $x^* \in X$ is said to be a *(globally) weakly efficient solution* of the BCP if there does not exist another $x^* \in X$ for which $f_i(x) < f_i(x^*)$, for both i = 1, 2. The image $f(x^*)$ of an (weakly) efficient solution x^* in the objective space is called a *(weakly) nondominated solution*. The set of all nondominated solutions is denoted by Y_E.

In general, the nondominated set of BCPs may be a convex or nonconvex curve, and connected or disconnected, i.e., composed of two curves with empty intersection. A connected set was defined by Bitran and Magnanti (1979) while TenHuisen and Wiecek (1996) applied that concept to define the disconnected nondominated set for BCPs and whose definition we use.

We shall treat the BCP with the weighted Tchebycheff approach in order to find its efficient solutions. We assume that each objective function f_i, i = 1,2, achieves a finite global minimum over the feasible set. Let $w_i \geq 0$, i = 1, 2, be the weights and u_i^*, i = 1, 2, be the elements of the utopia point defined as $u_i^* = \{\min f_i(x) - \delta_i,\ x \in X,\ \delta_i \geq 0\}$, i = 1, 2. Since in the sequel we require $w_i > 0$, i = 1, 2, we need $\delta_i > 0$, i = 1, 2, to generate efficient solutions being individual constrained minima of the objective functions. To find other efficient points we may use $\delta_i = 0$, i = 1, 2.

Now consider the problem

$$\min_{x \in X} \max_{i=1,2} \{\, w_i(\, f_i(x) - u_i^*) \,\}, \tag{2}$$

where $w_1 + w_2 = 1$. Program (2) may be written in an alternative form as the following

β-problem

$$\begin{array}{lll} \text{minimize} & \beta & \\ \text{subject to} & w_i(f_i(x) - u_i^*) \le \beta, \ i = 1, 2 & (3) \\ & x \in X, & \end{array}$$

where β is a nonnegative variable. We assume that program (3) has an optimal solution for every $w_i \ge 0$, $i = 1, 2$, at which at least one of the inequality constraints is always binding.

A relationship between the weighted Tchebycheff approach represented by problem (3) and its quadratic Lagrangian dual problem was established by Tind and Wiecek (1998). We now briefly review this duality result in the context of the BCP.

Theorem 1. (Tind and Wiecek, 1998)
Let program (3) related to the original BCP satisfy the quadratic growth condition and be stable of degree 2. Let x be an efficient point of the BCP generated by the weighted Tchebycheff scalarization with some ideal point u^* and some weights $w_i > 0$, $i = 1, 2$. The same efficient point is an optimal solution of the quadratic weighted-sum scalarization

$$\begin{array}{lll} \text{minimize} & q(f(x)) = f(x)^T Q f(x) + p^T f(x) + c & (4) \\ \text{subject to} & x \in X, & \end{array}$$

where $q(\cdot)$ is a real-valued function, Q is a symmetric m x m matrix, $p \in R^m$, and m is the number of the binding inequality constraints in program (3) at this efficient point.

In particular, if m = 2, then the parameters of program (4) assume the following form: Q is a symmetric 2×2 matrix

$$Q = (1/2)\,\alpha \begin{bmatrix} w_1^2 & -w_1 w_2 \\ -w_1 w_2 & w_2^2 \end{bmatrix}, \qquad (5)$$

$p = [p_1, p_2]^T \in R^2$,

$$p = \begin{bmatrix} -\alpha w_1{}^2 u_1{}^* + \alpha w_1 w_2 u_2{}^* + w_1 y_1 \\ -\alpha w_2{}^2 u_2{}^* + \alpha w_1 w_2 u_1{}^* + w_2 y_2 \end{bmatrix}, \qquad (6)$$

and c is a scalar,

$$
\begin{aligned}
c &= (u^*)^T Q\, u^* + w_1 y_1 u_1^* + w_2 y_2 u_2^* \\
&= 0.5\, \alpha\, (w_1 u_1^* - w_2 u_2^*)^2 + w_1 y_1 u_1^* + w_2 y_2 u_2^* .
\end{aligned} \tag{7}
$$

The coefficient α and the vector $y \in R^2$ are the dual variables: y is entirely determined by the weights

$$
(y_1, y_2) = \left(\frac{w_2^2}{w_1^2 + w_2^2}, \frac{w_1^2}{w_1^2 + w_2^2} \right), \tag{8}
$$

while $\alpha > 0$ is sufficiently large. For convenience, the function q(f) can be also written as:

$$
q(f) = 0.5\alpha(w_1 f_1 - w_2 f_2)^2 + p_1 f_1 + p_2 f_2 + c. \tag{9}
$$

It can be shown that the matrix Q is positive semi-definite so that program (4) involves the minimization of a convex quadratic function of the objective functions over the decision space.

If one of the inequality constraints in program (3) is nonbinding, the related objective function does not contribute to determining the nondominated point nor to the trade-offs with the other objective function and consequently, it is not present in the quadratic program. To make both objective functions present in the quadratic program, one can find another set of weights for which both constraints are binding and for these new weights derive the quadratic program of full size. For the discussion on the choice of appropriate weights see Steuer (1986).

The quadratic growth condition requires that the quadratic Lagrangian dual problem of program (3) be feasible. With this condition satisfied, stability of degree 2 is necessary and sufficient for the duality to hold between program (3) and the dual. For a more detailed discussion on these conditions the reader is referred to Rockafellar (1974).

Consider the following example as illustration of these results. Let BCP be given as

$$\begin{aligned} &\text{minimize} && f(x) = [\, x, -x^2 + 4]^T && (10) \\ &\text{subject to} && x \in [0, 2]. \end{aligned}$$

Taking $\delta_i = 0$, i = 1, 2, we get $u_i^* = 0$, i = 1, 2, and for $w_1 = 2/3$ and $w_2 = 1/3$ program (3) yields the efficient solution x = 1.236 and the nodominated outcome f(x) = [1.236, 2.472]. According to Theorem 1, the same efficient point is an optimal solution of the following program

$$\begin{aligned} &\text{minimize} && q(x) = 0.5\alpha(\,(2/3)\,x - (1/3)(-x^2 + 4)\,)^2 + 0.133x + 0.267(-x^2 + 4) \\ & && && (11) \\ &\text{subject to} && x \in [0, 2], \end{aligned}$$

where $\alpha > 0$ and sufficiently large. In Section 3 we show how this unknown parameter can be found.

3. Local Approximation

We now present a procedure to locally approximate the nondominated set of the BCP by means of the quadratic function q(f) (program (9)).

Let $\bar{f}$, referred to as a candidate point, be a nondominated solution of the BCP generated by the weighted Tchebycheff scalarization for given utopia point and weights. According to Theorem 1, the family of curves $q(f) = q(\bar{f})$ is a family of quadratic level curves supporting the nondominated set at the point $\bar{f}$. Observe that as the parameter α is positive, the quadratic level curves are concave and their steepness is controlled by its magnitude. The condition of stability of degree 2 guarantees that the curvature of the nondominated curve in a neighborhood of $\bar{f}$ is such that this curve can be supported at $\bar{f}$ by a quadratic function and the corresponding quadratic weighted-sums scalarization can be constructed.

In general, the curvature of an approximating function should follow the curvature of the nondominated set in a vicinity of the point of interest. Consider first a locally concave nondominated curve. In this case, only the steepness of the quadratic curve yielded by (9) has to be adjusted so that this curve approximates the nondominated set well. While using the quadratic curve (9) to approximate a locally convex nondominated curve, its curvature as well as the steepness are required to be adjusted. In any case, the adjustment is entirely related to the unknown parameter α that should now be found.

In order to determine the value of α so that the related level curve in the family $q(f) = q(\bar{f})$ best approximates the nondominated set in a neighborhood of $\bar{f}$, we first sample the nondominated set and find a few nondominated points in a neighborhood of the candidate point.

Let $\bar{f}$ be the candidate nondominated solution obtained by the weighted Tchebycheff scalarization. Let $f^1, f^2, \ldots, f^K$ be other nondominated solutions located in a neighborhood of $\bar{f}$ and obtained by solving K ε-constraint problems and formulated as (Haimes and Chankong, 1983):

$$
\begin{aligned}
&\text{minimize} && f_2(x) \\
&\text{subject to} && f_1(x) \leq \bar{f}_1 + \varepsilon_k \\
& && x \in X,
\end{aligned}
\tag{12}
$$

where $|\varepsilon_k| \leq \varepsilon$, $k = 1, 2, ..., K$, $\varepsilon_k \in R$, and $\varepsilon > 0$ determines the neighborhood range of f_1 at the candidate solution. The inequality constraint of problem (12) is referred to as the ε_k–constraint and the problem is denoted as $P_2(\varepsilon_k)$. If x^k is a unique optimal solution of problem (12), then x^k is efficient and $f(x^k)$ is nondominated. Additionally, solving several problems of type (12) for different right-hand-side values, and, at optimality, obtaining the inequality constraint active and then inactive, is an evidence of a disconnected nondominated set, which we prove below.

Theorem 2.

Let $\varepsilon_k > \varepsilon_{k+1}$, where k is a positive integer. Let x^k and x^{k+1} be unique optimal solutions of $P_2(\varepsilon_k)$ and $P_2(\varepsilon_{k+1})$, respectively. If the ε_k-constraint is active at optimality of $P_2(\varepsilon_k)$ and the ε_{k+1}-constraint is not active at optimality of $P_2(\varepsilon_{k+1})$, then the nondominated set Y_E is disconnected.

<u>Proof</u>: Since x^k and x^{k+1} are unique optimal solutions of $P_2(\varepsilon_k)$ and $P_2(\varepsilon_{k+1})$, respectively, they are both efficient solutions of the BCP and the corresponding points $f(x^k) = [f_1(x^k), f_2(x^k)]$, where $f_1(x^k) = \bar{f}_1 + \varepsilon_k$, and $f(x^{k+1}) = [f_1(x^{k+1}), f_2(x^{k+1})]$, where $f_1(x^{k+1}) < \bar{f}_1 + \varepsilon_{k+1}$,

are nondominated. Also, there is no nondominated point whose first component f_1 is equal to $\bar{f}_1 + \varepsilon_{k+1}$.

Define a line $L = \{(f_1, f_2): f_1 = \bar{f}_1 + \varepsilon_{k+1}\}$ and the related halfplanes $L_1 = \{(f_1, f_2): f_1 \geq \bar{f}_1 + \varepsilon_{k+1}\}$ and $L_2 = \{(f_1, f_2): f_1 \leq \bar{f}_1 + \varepsilon_{k+1}\}$ so that $Y_E \subset L_1 \cup L_2$. Since $f(x^k) \in L_1$ and $f(x^{k+1}) \in L_2$, then $Y_E \cap L_1 \neq \emptyset$ and $Y_E \cap L_2 \neq \emptyset$, however $L_1 \cap L_2 \cap Y_E = \emptyset$. Consequently, Y_E is disconnected. □

Let $\bar{f}^k$, k = 1, 2, …, K denote approximate nondominated solutions corresponding to the true nondominated solutions f^k, k = 1, …, K. Given these two groups of points, we assume that the approximating function takes on the same values at the approximate points as it does at the nondominated points. We therefore define a real-valued function

$$\varphi(\alpha) = \sum_{k=1}^{K} (q(\bar{f}^k) - q(f^k))^2, \tag{13}$$

whose minimization yields the desired value of the parameter α. Since $q(\bar{f}^k) = q(\bar{f})$ for all k = 1, 2, … , K, we solve

$$\begin{aligned} &\text{minimize} \qquad \varphi(\alpha) = \sum_{k=1}^{K} (q(\bar{f}) - q(f^k))^2, \\ &\text{subject to} \qquad \alpha \in R. \end{aligned} \tag{14}$$

The optimal solution of this problem is yielded by the equation

$$\frac{\partial \varphi(\alpha)}{\partial \alpha} = 0. \tag{15}$$

We get

$$\alpha^* = \frac{\sum_{k=1}^{K} [d_2(\bar{f}) - d_2(f^k)][d_1(\bar{f}) - d_1(f^k)]}{\sum_{k=1}^{K} [d_1(\bar{f}) - d_1(f^k)]^2}, \tag{16}$$

where

$$d_1(f) = 0.5(w_1 f_1 - w_2 f_2)^2 + w_1 w_2 u_2^* f_1 + w_1 w_2 u_1^* f_2 - w_1^2 u_1^* f_1 - w_2^2 u_2^* f_2 \tag{17}$$

$$d_2(f) = w_1 y_1 f_1 + w_2 y_2 f_2. \tag{18}$$

The optimal value α^* given by (16) may be positive, negative, or zero. If it is positive, the corresponding quadratic function q(f) satisfies the relationship of Theorem 1 and the quadratic approximation is locally concave around the candidate point. Otherwise, this relationship does not hold, however the corresponding function can be used to approximate the nondominated set. When $\alpha^* = 0$, the quadratic term in (9) vanishes which implies that the approximation is locally linear, and when $\alpha^* < 0$, it is locally convex.

The approach presented above yields the approximating equation

$$AF(f) - AF(\bar{f}) = 0, \tag{19}$$

in a neighborhood of $\bar{f}$, where

$$AF(f_1, f_2) = 0.5\alpha^*(w_1 f_1 - w_2 f_2)^2 + p_1 f_1 + p_2 f_2 + c. \tag{20}$$

We can calculate the approximation error $e(f^k)$ at an auxiliary nondominated point f^k as follows:

$$e(f^k) = \frac{\left|AF(f^k) - AF(\bar{f})\right|}{AF(\bar{f})} \times 100\%, \tag{21}$$

and the maximal approximation error as

$$e_{max} = \max_{k=1,2,\ldots,K} e(f^k). \tag{22}$$

Due to equation (19), the local approximation of the nondominated set is available in a closed-form. This approximation makes use of the utopia point inherent to the BCP and the weights selected by the decision maker according to his/her preferences and opinion on the importance of the two objective functions. The obtaining of equation (19) does not put an additional burden on the decision maker but relies solely on the information needed for finding the candidate solution. The same information is used to quadraticly weigh the objective functions and provide the approximation based on the preferences the decision maker is conscious of. Another distinct feature of this approach is that the problem of finding the approximating curve is reduced to single-variable optimization due to searching for an optimal α.

Consider again problem (10) and the candidate solution $\bar{f}$ = [1.236, 2.472]. Let f^1 = [1, 3] be a nondominated solution yielded by program (12). Solving problem (14) for K = 1, we find α^* = -0.653 and the approximating equation (19) becomes

$$-0.327\left((2/3)f_1 - (1/3)f_2 \right)^2 + 0.133\, f_1 + 0.267\, f_2 - 0.824 = 0.$$

4. An approximation algorithm

In this section, an algorithm to locally approximate the nondominated set of BCPs is presented. The algorithm can be applied interactively to adjust the interval of approximation in the case the interval initially specified is too big to meet the error limit or in the case the nondominated set is disconnected.

The outline of the algorithm *local approximation* is as follows. The input includes the original BCP and its ideal point u^*. The algorithm consists of the initialization step and the main step. During the initialization, a nondominated candidate point $\bar{f}$ is found by solving the following modification of program (3) for some positive weights w_1 and w_2:

$$\begin{array}{ll} \text{minimize} & [\beta, \sum_{i=1}^{2} (f_i(x) - u_i^*)] \\ \text{subject to} & w_i(f_i(x) - u_i^*) \le \beta,\ i = 1, 2 \\ & x \in X, \end{array} \tag{23}$$

which is referred to as the lexicographic weighted Tchebycheff scalarization (Steuer, 1986). If at optimality of program (23), m = 2, one may proceed to the main step. Otherwise, the weight corresponding to the inactive constraint has to be adjusted to make this constraint active at the nondominated point found, as was discussed in Section 2. Given $\bar{f}$, the interval of approximation $[\varepsilon_{min}, \varepsilon_{max}]$ is selected so that $\bar{f}_1 \in [\varepsilon_{min}, \varepsilon_{max}]$ and problem (12) solved for any $\bar{f}_1 + \varepsilon_k \in [\varepsilon_{min}, \varepsilon_{max}]$ is feasible. The approximation range ε is equal to $\varepsilon_{max} - \varepsilon_{min}$. One also chooses L, the number of auxiliary nondominated points to be used, and σ, the error limit. While the weights may be viewed as subjective parameters representing the importance of the two criteria for the decision maker, the other input parameters including $[\varepsilon_{min}, \varepsilon_{max}]$, L, and σ are objective in the sense that they only affect computational complexity of the algorithm and the quality of the resulting approximation.

The main step of the algorithm calls the procedure *nondominated* that returns K auxiliary nondominated points, $K \leq L$, and identifies intervals of f_1 within which there are no f_1-components of nondominated points. If such intervals have not been identified, the approximating equation (19) is found for the approximation interval originally selected, and if the maximal error is acceptable the algorithm is completed. Otherwise, a new smaller interval is chosen within which the nondominated set is examined or the set is disconnected and the original approximation interval is modified to exclude the identified intervals.

In the procedure *nondominated*, the problem $P_2(\varepsilon_k)$ is solved. If its optimal solution x^k is unique, then this solution is efficient. Additionally, if the ε_k-constraint is also active, a new $\varepsilon_{k+1} < \varepsilon_k$ is chosen and the procedure continues. Otherwise, there are no nondominated points whose first component f_1 is in the interval $(f_1(x^k), \varepsilon_k]$ and the index M tracking the number of such intervals is increased by one. If x^k is not unique, then x^k is weakly efficient and the procedure *weakly nondominated* is called. In this procedure, the following problem denoted as $P_1(\varepsilon_k, f_2(x^k))$ is solved:

$$\begin{aligned} &\text{minimize} && f_1(x) \\ &\text{subject to} && f_1(x) \leq \varepsilon_k \\ & && f_2(x) = f_2(x^k) \\ & && x \in X. \end{aligned} \tag{24}$$

If x^k is an optimal solution of the problem $P_1(\varepsilon_k, f_2(x^k))$, then this solution is efficient and there are no nondominated points whose first component f_1 is in the interval $(f_1(x^k), \varepsilon_k]$. The index W tracking the number of such intervals is increased by one, a new $\varepsilon_{k+1} < \varepsilon_k$ is chosen and the procedure continues.

Below we present a pseudo-code of the algorithm and two procedures. The issue of choosing ε_{k+1} is not resolved since the decision maker may arbitrarily decide whether this choice should be made interactively or set up automatically in the algorithm. The interactive option offers flexibility in locating the auxiliary nondominated points while the other option offers convenience.

algorithm local approximation

begin *initialization*

choose weights w_1, $w_2 > 0$

solve program (23); let $\bar{f}$ be the nondominated solution found such that m = 2

choose $[\varepsilon_{min},\varepsilon_{max}]$, L, σ

end

begin *main step*

apply *nondominated*

if M = 0, **then** calculate α^* using (16), derive AF(f) according to (20),

calculate AF($\bar{f}$), and calculate e_{max}

if $e_{max} < \sigma$ **then** equation (19) is a good approximation of Y_E within the current interval

else choose $[\varepsilon_{min},\varepsilon_{max}]^{new} : [\varepsilon_{min},\varepsilon_{max}]^{new} \subset [\varepsilon_{min},\varepsilon_{max}]$

set $[\varepsilon_{min},\varepsilon_{max}] := [\varepsilon_{min},\varepsilon_{max}]^{new}$

else modify $[\varepsilon_{min},\varepsilon_{max}]$

end

procedure *nondominated*

begin

set k : = 1, K : = 0, M : = 0, W : = 0

set $\varepsilon_k := \varepsilon_{max}$

while $k \leq L$ **do**

begin

solve $P_2(\varepsilon_k)$; let x^k be an optimal solution

if x^k is not unique **then** *weakly nondominated*

else K : = K + 1

if ε_k–constraint is active **then** choose $\varepsilon_{k+1} \in [\varepsilon_{min}, \varepsilon_k)$, set k : = k + 1

else no nondominated points whose $f_1 \in (f_1(x^k), \varepsilon_k]$

set M : = M + 1

choose $\varepsilon_{k+1} \in [\varepsilon_{min}, f_1(x^k))$, and set k : = k + 1

end

end

procedure *weakly nondominated*

begin

set W : = W + 1

solve $P_1(\varepsilon_k, f_2(x^k))$; let x^k be an optimal solution, set K : = K + 1

no nondominated points whose $f_1 \in (f_1(x^k), \varepsilon_k]$

choose $\varepsilon_{k+1} \in [\varepsilon_{min}, f_1(x^k))$, set k : = k + 1

end

In general, the approximation error should be smaller when more auxiliary nondominated solutions are used to find the optimal α. Since obtaining one solution requires solving one ε-constraint problem, the computational cost could be high if too many nondominated solutions are to be found. We propose to use one or two auxiliary points in addition to the candidate point $\bar{f}$. When K = 1, expression (16) simplifies to:

$$\alpha^* = \frac{d_2(\bar{f}) - d_2(f^1)}{d_1(\bar{f}) - d_1(f^1)} \tag{25}$$

and the proposed algorithm becomes a two-point algorithm since two nondominated points are used while for K = 2 this expression yields

$$\alpha^* = \frac{\sum_{k=1}^{2}[d_2(\bar{f}) - d_2(f^k)][d_1(\bar{f}) - d_1(f^k)]}{\sum_{k=1}^{2}[d_1(\bar{f}) - d_1(f^k)]^2} \tag{26}$$

and the proposed algorithm becomes a three-point algorithm since three nondominated points are used.

It is a well-known fact that all nondominated solutions of the BCP can be found by means of the lexicographic weighted-Tchebycheff scalarization (Steuer, 1986) independently of the curvature of the nondominated set. A nondominated solution can be generated even if it is located at a point where the nondominated curve is not differentiable and even if additionally this solution is improperly nondominated (in the sense of Geoffrion (see Geoffrion, 1968)). The assumptions of Theorem 1 have to hold only for the duality results and are not related to the ability of finding a nondominated solution. Since we use the results of Theorem 1 to develop a local approximation of the nondominated set rather than to solve the quadratic weighted-sums program, we may apply Theorem 1 at every nondominated point and the approximation error will inform the user on the accuracy of the approximation.

Given the quadratic local approximation of the nondominated set in a neighborhood of a candidate point, one may choose several candidate points and find piecewise quadratic approximation of a part or the whole nondominated set.

5. Example

We illustrate the approximation algorithm on the following BCP:

$$\begin{aligned} &\text{minimize} && f(x) = [f_1(x), f_2(x)]^T \\ &\text{subject to} && g(x) = - x_1 - x_2 + 0.1 \le 0 \\ &&& 0 \le x_1 \le 10 \\ &&& 0 \le x_2 \le 10, \end{aligned} \tag{27}$$

where

$$f_1(x) = 10.0(x_1 - 2.0)^4 + 10.0(x_1 - 2.0)^3 + 10.0(x_2 - 2.0)^4 + 10.0(x_2 - 2.0)^3 + 10.0$$
$$f_2(x) = (x_1 - 3.0)^2 + (x_2 - 3.0)^2 + 10.0.$$

For this example, the utopia point $u^* = (6.89066, 9.0)$. We choose weights $(w_1, w_2) = (0.77, 0.23)$ and find the efficient solution $\bar{x} = (1.46872, 1.46872)$ and the corresponding nondominated candidate point $\bar{f} = (\bar{f}_1, \bar{f}_2) = (8.594, 14.690)$.

Tables 1 and 2 list the coefficients of the approximating functions and the related errors found by the algorithm for K = 1 and K = 2 and different ranges of approximation. For every range, the choice of ε_k, k = 1, 2,..., K was performed interactively. Although Table 1 reports findings for K = 1, two points, f^1 and f^2, are used to calculate the error (f^1 is found by setting $\varepsilon_k = \varepsilon_1$ in program (12) while f^2 is found by setting $\varepsilon_k = \varepsilon_2$, where $\varepsilon_2 = \varepsilon_1/2$). From the tables we observe that the approximation is the best in a close neighborhood of the candidate point and that the magnitude of α^* depends on the approximation range. Figure 1 depicts the approximating functions listed in Tables 1 and 2. The points marked by the symbol "+" in these figures represent the nodominated set of this example. The lighter dashed curve represents the quadratic function obtained by the two-point algorithm while the darker curve represents the quadratic function obtained by the three-point algorithm. The optimal values of α found by the two-point algorithm and the three-point algorithm are denoted $\alpha^{(2)}$ and $\alpha^{(3)}$, respectively.

Observe that for the larger approximation ranges, the quadratic functions determined by the two-point algorithm are different from the ones obtained by the three-point algorithm (see pictures (a) and (b) in Figure 1). However for the smaller ranges, the quadratic functions obtained by the two-point algorithm are very close to the ones obtained by the three-point algorithm (see pictures (c), (d), (e) and (f) in Figure 1). The approximation error of the two-point algorithm (K = 1) and the three-point algorithm (K = 2) over the different ranges is further illustrated in Figure 2. Clearly, for the smaller approximation ranges, the error yielded by the two algorithms is almost the same. In general, the two-point algorithm could be used for obtaining an acceptable approximating function for smaller ranges while for larger ranges the three-point algorithm is recommended.

Table 1. Approximation Errors
($w_1 = 0.77, w_2 = 0.23$, K = 1)

Range ε	p_1	p_2	c	α^*	$e(f^1)$	$e(f^2)$
5.0	-0.0588	0.2476	2.5911	0.0489	0.00%	7.65930%
2.5	-0.3481	0.3340	3.1989	0.1650	0.00%	2.44888%
1.2	-0.7241	0.4463	3.9889	0.3159	0.00%	0.86765%
0.6	-1.3880	0.6646	5.3841	0.5824	0.00%	0.47028%
0.3	-2.7794	1.0602	8.3077	1.1408	0.00%	0.24089%
0.15	-5.5587	1.8904	14.147	2.2563	0.00%	0.19225%

Table 2. Approximation Errors
($w_1 = 0.77$, $w_2 = 0.23$, K = 2)

Range ε	p_1	p_2	c	α^*	$e(f^1)$	$e(f^2)$
5.0	-0.0888	0.2565	2.6542	0.0609	2.3366%	6.86387%
2.5	-0.3670	0.3396	3.2388	0.1726	0.5019%	2.12642%
1.2	-0.7575	0.4563	4.0592	0.3293	0.1897%	0.82389%
0.6	-1.4753	0.6706	5.5675	0.6174	0.1140%	0.44079%
0.3	-2.9510	1.1115	8.6682	1.2097	0.0579%	0.22608%
0.15	-5.8925	1.9901	14.849	2.3903	0.0287%	0.11940%

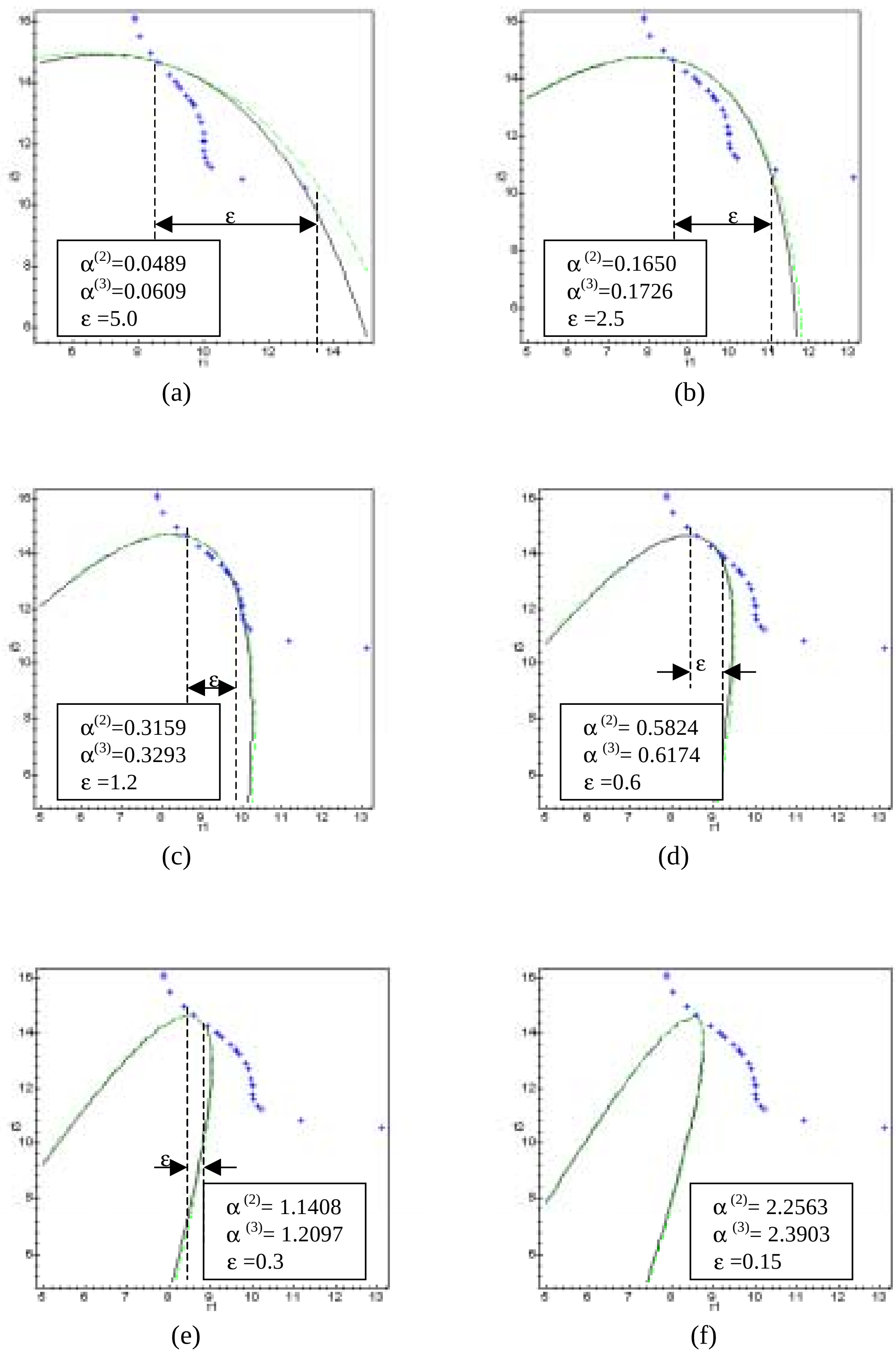

Figure 1. Local Approximation of the Nondominated Set
($w_1 = 0.77$, $w_2 = 0.23$)

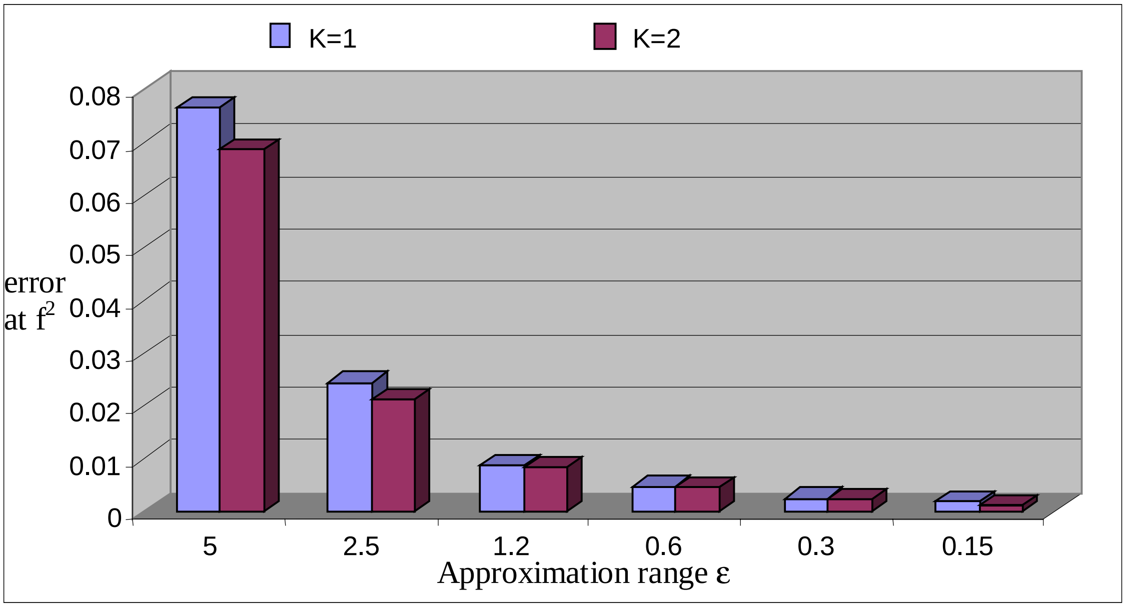

Figure 2. Approximation Error Comparison

Assume that for the candidate solution $\bar{f}$ = (8.594, 14.690), the approximation error obtained by the three-point algorithm is acceptable in the range $\varepsilon = 1.2$ for which $\alpha^* = 0.3293$. The positive sign of α^* is the evidence that the approximating function is concave within the chosen range. Table 3 shows the two auxiliary points, f^1 and f^2, found by the ε-constraint method for $\varepsilon_1 = 1.2$ and $\varepsilon_2 = 0.6$, respectively. The resulting approximating function, denoted by AF_1, is depicted in Figure 3 where the candidate point is marked as A^1. The information about this function is collected in Table 4 in the column of AF_1. The first two rows of this table show the ranges of the two objective functions for which this approximation is satisfactory. Rows 3 through 7 show the coefficients of the approximating function.

The decision maker may now choose to approximate the nondominated set for the candidate point $A^2 = A^1$, but in the range $\varepsilon = 0.6$ with $\varepsilon_1 = -0.3$ and $\varepsilon_2 = -0.6$. The three-point algorithm results in the approximating function AF_2 depicted again in Figure 3. The second row of Table 3 and the AF_2 column of Table 4 give the details of this approximation. Observe that optimal value of α is negative as we now produce a convex approximating curve.

In order to approximate the whole nondominated set, the decision maker may now choose another pair of weights, say (0.4, 0.6), yielding the candidate point $A^3 = (10.239, 11.234)$.

Table 3. Constructing Approximating Functions

Candidate solution $\bar{f}$	Range ε	Auxiliary points				α^*	Appro. Function
		ε_1	f^1	ε_2	f^2		
A^1 (8.594, 14.690)	1.2	1.2	(9.794, 13.066)	0.6	(9.187, 13.976)	0.3293	AF_1
A^2 (8.594, 14.690)	0.6	-0.3	(8.294, 15.088)	-0.6	(7.994, 15.628)	-0.7316	AF_2
A^3 (10.239,11.234)	3.245	2.8	(13.039,10.563)	-0.445	(9.794, 13.066)	-0.4630	AF_3

Table 4. Approximation Summary

Approximating Function	AF_1	AF_2	AF_3
range for f_1	[8.594, 9.794]	[7.994, 8.594]	[9.794, 13.039]
range for f_2	[14.690, 13.066]	[15.628, 14.690]	[13.066, 10.563]
(w_1, w_2)	(0.77, 0.23)	(0.77, 0.23)	(0.4, 0.6)
α^*	0.32934	-0.73162	-0.46298
(p_1, p_2)	(-0.7575,0.4563)	(1.8860, -0.3333)	(-0.2127,0.9190)
c	4.05922	-1.49513	1.95176
$AF(A) = AF(\bar{f})$	5.97899	5.97899	8.47927

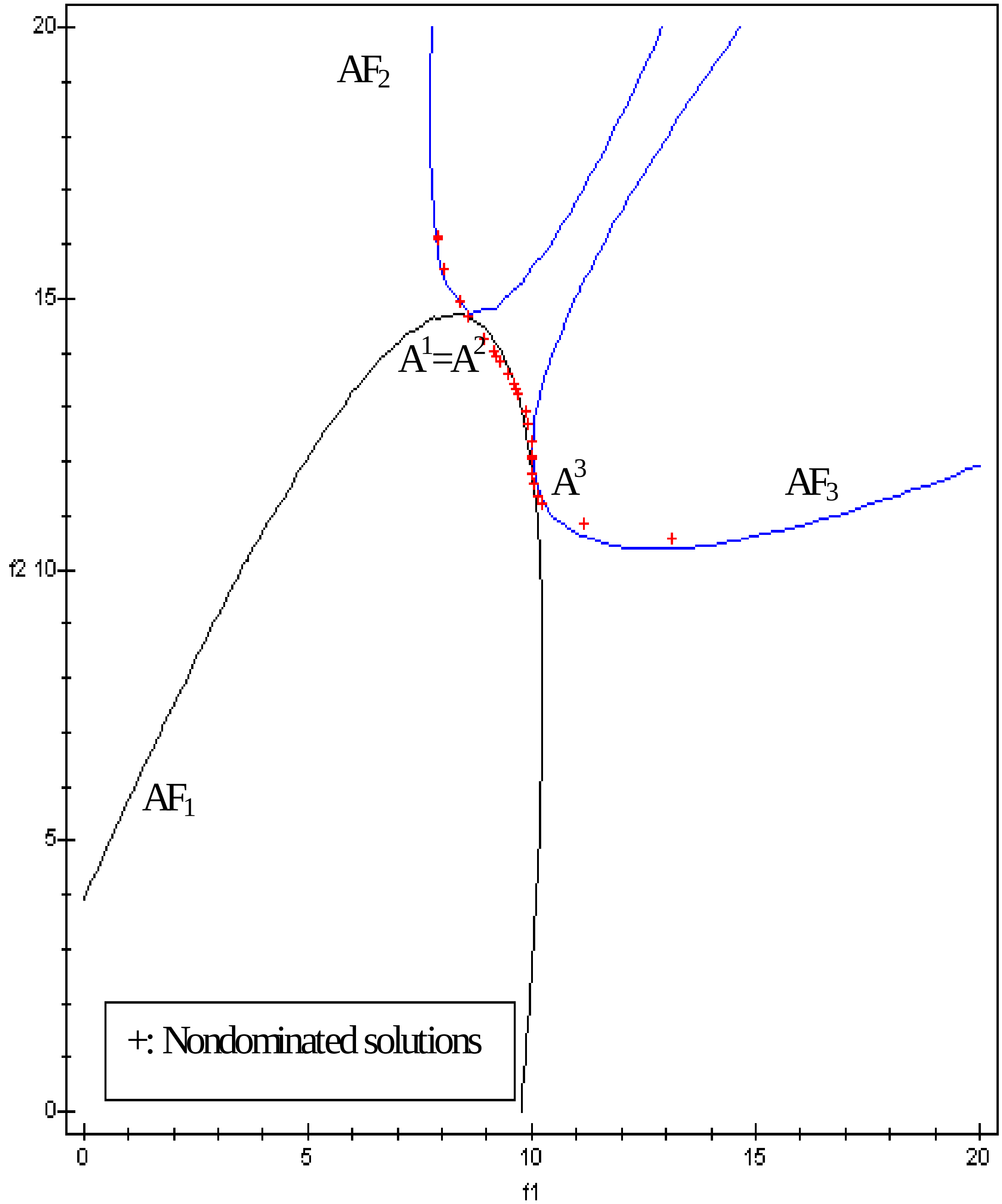

Figure 3. Approximation of the Nondominated Set

Within the approximation range $\varepsilon = 3.245$, two auxiliary points, f^1 and f^2, are found for $\varepsilon_1 = 2.8$ and $\varepsilon_2 = -0.445$, respectively. The choice of ε_2 ensures continuity of the approximation as point f^2 found for the candidate point A^3 is the same as point f^1 found for the candidate point A^1. Figure 3 as well as Tables 3 and 4 give the related information.

Accuracy of the approximation is evaluated in Table 5. The maximum approximation error calculated at the auxiliary points is 1.929% while the minimum value of $\varphi(\alpha)$ is 0.04008. We may conclude that for this example problem the quadratic approximation obtained by the three-point algorithm has good accuracy. Clearly, more numerical experiments are needed to evaluate the effectiveness of the approximation for other classes of problems.

The functions AF_1, AF_2, and AF_3 give a piecewise quadratic approximation of the nondominated curve spanned between the points (7.994, 15.628) and (13.039, 10.539). Comparing these points to the utopia point, we observe that a big portion of the nondominated set has been approximated.

Table 5. Accuracy of Approximation

Approximating Function	AF_1	AF_2	AF_3
A	(8.594, 14.690)	(8.594, 14.690)	(10.239, 11.234)
f^1	(9.794, 13.066)	(8.294, 15.088)	(13.039, 10.563)
f^2	(9.187, 13.976)	(7.994, 15.628)	(9.794, 13.066)
$AF(f^1)$	5.99033	6.00675	8.31572
$AF(f^2)$	5.92972	5.97276	8.59475
$e(f^1)$	0.189%	0.104%	1.929%
$e(f^2)$	0.824%	0.464%	1.362%
$\varphi(\alpha^*)$	0.002556	0.000809	0.04008

6. Conclusions

In this paper, a procedure to approximate the nondominated set for general (continuous) bi-criteria programs (BCPs) is proposed. The approximation is developed locally in a neighborhood of a nondominated point of interest and is based on primal-dual relationships associated with the weighted-Tchebycheff scalarization of the original problem. The approximating function is quadratic and its structure is entirely determined by the primal-dual relationships. The BCP's utopia point and the weights selected by the decision maker determine almost all parameters of the function which makes the approximation comprehensive for the decision maker. The only parameter to be computed is uniquely determined by minimizing the differences between some selected true nondominated solutions and the corresponding approximate nondominated solutions. Approximation of the entire nondominated set may be achieved by concatenation of the local approximating curves. The procedure is additionally able to recognize whether the nondominated set is disconnected due to the existence or lack of weakly nondominated points.

Computational complexity of the proposed procedure entirely depends upon the complexity of single objective optimization problems that have to be solved to get a nondominated point of interest and a desired number of additional nondominated points. Therefore obtaining the approximation is computationally only as difficult as finding these nondominated points.

The derived approximating function facilitates the exploration of the nondominated set and the choice of a preferred nondominated solution of the BCP. This is extremely useful in practical decision making, for example, in engineering design where the information on design alternatives often provides additional knowledge of a system and therefore could help designers make a better decision. With our approach, the need of generating a large number of nondominated points is eliminated. This is practically significant for complex design problems that are computationally expensive.

Finally, the proposed procedure could also be extended to multiple objective programs since the primal-dual relationships hold for those problems too.

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